

# THE AGGREGATION EQUATION WITH NEWTONIAN POTENTIAL

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**ABSTRACT.** The viscous and inviscid aggregation equation with Newtonian potential models a number of different physical systems, and has close analogs in 2D incompressible fluid mechanics. We consider a slight generalization of these equations in the whole space, establishing well-posedness and spatial decay of the viscous equations, and obtaining the convergence of viscous solutions to the inviscid solution as the viscosity goes to zero.

## CONTENTS

|  |    |
|--|----|
| 1. Introduction  | 1  |
| 2. The viscous problem   | 3  |
| 2.1. The linear problem  | 4  |
| 2.2. The nonlinear problem   | 5  |
| 3. Spatial decay of viscous solutions  | 8  |
| 4. Total mass and infinite energy  | 10 |
| 5. The inviscid problem  | 13 |
| 6. The vanishing viscosity limit for $(GAG_\nu)$ for velocities in $L^2$     | 15 |
| 7. The vanishing viscosity limit for $(GAG_\nu)$ for velocities not in $L^2$ | 18 |
| 8. The vanishing viscosity limit in the $L^\infty$ -norm                     | 22 |
| 9. Concluding Remarks  | 25 |
| Acknowledgements   | 25 |
| References   | 25 |

## 1. INTRODUCTION

In this work, we study, on  $\mathbb{R}^d$ ,  $d \geq 2$ , the (viscous or inviscid) aggregation equation with Newtonian potential,

$$(AG_\nu) \quad \begin{cases} \partial_t \rho^\nu + \operatorname{div}(\rho^\nu \mathbf{v}^\nu) = \nu \Delta \rho^\nu, \\ \mathbf{v}^\nu = -\nabla \Phi * \rho^\nu, \\ \rho^\nu(0) = \rho_0. \end{cases}$$

Here,  $\nu \geq 0$  is the viscosity and  $\Phi$  is the fundamental solution of the Laplacian, or Newtonian potential (so  $\Delta \Phi = \delta$  and  $\operatorname{div} \mathbf{v}^\nu = -\rho^\nu$ ). The density is  $\rho^\nu$ , the velocity is  $\mathbf{v}^\nu$ , and  $\rho_0$  is the initial density.

Many variations on these equations are considered in the literature, primarily by using potentials other than the Newtonian or by using more general diffusive terms. We restrict our attention to the Newtonian potential with linear diffusion, for we will be concerned with analyzing the viscous ( $\nu > 0$ ) and inviscid ( $\nu = 0$ ) aggregation equation using techniques adapted from the study of 2D fluid mechanics.

The aggregation equation models many different physical problems. For the Newtonian potential, as in  $(AG_\nu)$ , this includes type-II superconductivity when  $\nu = 0$  (see [20] and the references therein) and chemotaxis, where  $(AG_\nu)$  for  $\nu > 0$  is a limiting case of the Keller-Segel system (see Section 5.2 of [21]), and has been extensively studied. In this context,  $\rho^\nu$  measures the density of cells (bacteria or cancer cells, for instance) and  $\mathbf{v}^\nu$  is the gradient of the concentration of a chemoattractant. References most closely related to the approach to the aggregation equation taken in this paper include [2, 3, 4, 14, 15, 20, 21].

We will, in fact, consider a slightly more general set of equations of the form

$$(GAG_\nu) \quad \begin{cases} \partial_t \rho^\nu + \mathbf{v}^\nu \cdot \nabla \rho^\nu = \sigma_2 (\rho^\nu)^2 + \nu \Delta \rho^\nu, \\ \mathbf{v}^\nu = \sigma_1 \nabla \Phi * \rho^\nu, \\ \rho^\nu(0) = \rho_0, \end{cases}$$

where  $\sigma_1, \sigma_2$  are constants with  $\sigma_1 \neq 0$ . When  $\sigma_1 = -1, \sigma_2 = 1$ ,  $(GAG_\nu)$  reduces to  $(AG_\nu)$ , since then  $\operatorname{div}(\rho^\nu \mathbf{v}^\nu) = \mathbf{v}^\nu \cdot \nabla \rho^\nu + \operatorname{div} \mathbf{v}^\nu \rho^\nu = \mathbf{v}^\nu \cdot \nabla \rho^\nu - (\rho^\nu)^2$ . We will study these equations in all of  $\mathbb{R}^d$ , though much of what we find extends naturally to a bounded domain given appropriate boundary conditions.

At least one other special case of  $(GAG_\nu)$  has been studied in the literature:  $(GAG_0)$  with  $\sigma_1 = -1, \sigma_2 = 0$  are derived from  $(AG_0)$  by making a transformation of variables in (1.6) of [2]. This transformation applies only in the special case of aggregation patch initial data (analogous to vortex patches for fluids) for  $(AG_0)$ . Although this transformation only works for aggregation patch initial data, the authors of [2] go on to use this special case of  $(GAG_0)$  throughout their analysis of aggregation patches. A general well-posedness result is not needed in [2] and hence not established there, but such a result was one of our motivations for studying the generalization of  $(AG_\nu)$  in  $(GAG_\nu)$ , the parameters  $\sigma_1, \sigma_2$  merely interpolating between  $(AG_0)$  and the equations studied in [2].

We will find establishing the existence of weak viscous solutions to  $(GAG_\nu)$  no more difficult than doing the same for  $(AG_\nu)$ , except for keeping track of the constants  $\sigma_1$  and  $\sigma_2$ . We give a proof of existence of weak solutions for  $\rho_0 \in L^1 \cap L^\infty$  in Section 2. The result we obtain, specifically a bound on the existence time, is suited to our needs in later sections, though much more is known about the existence time of solutions, at least for  $(AG_\nu)$  for nonnegative  $\rho_0$  (as summarized in Sections 5.2, 5.3 of [21]). Uniqueness for solutions to  $(GAG_\nu)$  when  $\sigma_1 + \sigma_2 = 0$  follows, even for  $\rho_0 \in BMO$ , using Yudovich's uniqueness argument (in the form in [24]) as proved in [1].

In Section 3 we bound the spatial decay of viscous solutions, bounds that will be required later in establishing the vanishing viscosity limit.

The varying effects of  $\sigma_1$  and  $\sigma_2$  begin to become apparent when we examine the behavior of the total mass of the density,  $m(\rho^\nu) := \int_{\mathbb{R}^d} \rho^\nu$ , in Section 4. We will find that  $m(\rho^\nu)$  is conserved only when  $\sigma_1 + \sigma_2 = 0$ .

The well-posedness of the inviscid equations,  $(GAG_0)$ , are the subject of Section 5.

In Section 6 we begin our analysis of the vanishing viscosity limit of solutions of  $(GAG_\nu)$  to a solution to  $(GAG_0)$  with the same initial data, showing that

$$(VV) : \quad \mathbf{v}^\nu \rightarrow \mathbf{v}^0 \text{ in } L^\infty(0, T; H^1), \quad \rho^\nu \rightarrow \rho^0 \text{ in } L^\infty(0, T; L^2) \text{ as } \nu \rightarrow 0.$$

When  $d = 3$ ,  $\mathbf{v}^\nu$  and  $\mathbf{v}^0$  both lie in  $L^2(\mathbb{R}^d)$ . When  $d = 2$ , this is no longer (in general) the case, the energies being infinite. When  $\sigma_1 + \sigma_2 = 0$ , however, because the total mass of the densities  $\rho^\nu$  and  $\rho^0$  are conserved over time, the infinite parts of the energies cancel, giving  $\mathbf{v}^\nu - \mathbf{v}^0 \in L^2(\mathbb{R}^2)$ . In both of the cases,  $d \geq 3$  or  $d = 2$  with  $\sigma_1 + \sigma_2 = 0$ , (VV) holds, as we show in Section 6.

In Section 7 we consider the remaining case where  $d = 2$  but  $\sigma_1 + \sigma_2 \neq 0$ . In this case, the total mass of the densities are not conserved over time, and the infinite parts of the energies do not cancel. We will nonetheless be able to isolate the infinite parts of the energy and use them to define a corrector,  $\theta^\nu$ , that lies in weak- $L^2$  and all higher  $L^p$  spaces, and show that in place of  $(VV)$ , we have

$$(VV)' : \quad \mathbf{v}^\nu - \mathbf{v}^0 - \theta^\nu \text{ in } L^\infty(0, T; H^1), \quad \rho^\nu \rightarrow \rho^0 \text{ in } L^\infty(0, T; L^2) \text{ as } \nu \rightarrow 0, \\ \theta^\nu \rightarrow 0 \text{ in } L^\infty([0, T]; C^k) \text{ for all } k \geq 0.$$

As can be seen from  $(VV)$ ,  $(VV)'$ , both the velocity and density converge strongly in the vanishing viscosity limit. Indeed, the arguments in Sections 6 and 7 involve showing the simultaneous convergence of both the velocities and the densities.

In Section 8, we use the results from Sections 6 and 7, along with uniform bounds in viscosity on Holder norms of solutions to  $(GAG_\nu)$ , to prove that the vanishing viscosity limit holds in the  $L^\infty$ -norm of the density.

We close by stating a few conventions and making one definition.

We follow the convention that  $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^d)}$ . We write  $\langle \cdot, \cdot \rangle$  for the  $L^2$ -inner product and  $(\cdot, \cdot)$  for the pairing in the duality between  $H^1$  and  $H^{-1}$ .

Our proofs of the vanishing viscosity limit yield rates of convergence in both viscosity and time, but it will be unwieldy to keep track of the specific dependence on time. Hence, we will use the convention that

$C_0(t)$  is a positive, continuous, nondecreasing function of  $t \in [0, \infty)$ .

We make the convention that  $C(a_1, \dots, a_n)$  stands for a continuous function from  $[0, \infty)^n \rightarrow [0, \infty)$  that is nondecreasing in each of its arguments. We use  $C(a_1, \dots, a_n)$  in the context of a constant that depends on the parameters  $a_1, \dots, a_n$ , where the exact form of the constant is unimportant.

We will find various uses for the following cutoff function:

**Definition 1.1.** *Let  $a$  be a radially symmetric function in  $C^\infty(\mathbb{R}^d)$  supported in  $B_2(0)$  with  $a \equiv 1$  on  $B_1(0)$  and with  $a(x)$  nonincreasing in  $|x|$ . For any  $R \geq 1$  define  $a_R(\cdot) = a(\cdot/R)$ . Note that for any fixed  $x \in \mathbb{R}^d$ ,  $a_R(x)$  is nondecreasing in  $R$ .*

For any  $p_1, p_2 \in [1, \infty]$ , we define  $\|f\|_{L^{p_1} \cap L^{p_2}} = \|f\|_{L^{p_1}} + \|f\|_{L^{p_2}}$ .

## 2. THE VISCOUS PROBLEM

Definition 2.1 gives our definition of a weak solution to the aggregation equation. This definition applies for both viscous and inviscid solutions.

**Definition 2.1.** *Let  $\nu \geq 0$  and  $\rho_0 \in L^1 \cap L^\infty$ . We say that  $\rho^\nu$  is a weak solution to the aggregations equations  $(GAG_\nu)$  on the interval  $[0, T]$  with initial density  $\rho_0$  if  $\rho^\nu(0) = \rho_0$ ,*

$$\begin{aligned} \rho^\nu &\in L^\infty(0, T; L^1 \cap L^\infty) \cap C([0, T]; L^2), \\ \partial_t \rho^\nu &\in L^2(0, T; H^{-1}), \\ \rho^\nu &\in L^2(0, T; H^1) \quad \text{if } \nu > 0, \end{aligned} \tag{2.1}$$

with

$$\int_0^T \int_{\mathbb{R}^d} (\rho^\nu \partial_t \varphi + \rho^\nu \mathbf{v}^\nu \cdot \nabla \varphi + (\sigma_1 + \sigma_2)(\rho^\nu)^2 \varphi - \nu \nabla \rho^\nu \cdot \nabla \varphi) = 0 \quad (2.2)$$

for all  $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^d)$ .

**Remark 2.2.** By the initial condition  $\rho^\nu(0) = \rho_0$  we mean that  $\rho^\nu(t) \rightarrow \rho_0$  in  $L^2$  as  $t \rightarrow 0^+$ , which makes sense because  $\rho^\nu \in C([0, T]; L^2)$ .

In this section we treat weak solutions to  $(GAG_\nu)$  for  $\nu > 0$ . In Section 5 we treat the case  $\nu = 0$ .

Define the total mass of  $f \in L^1(\mathbb{R}^d)$  by

$$m(f) := \int_{\mathbb{R}^d} f. \quad (2.3)$$

We consider first in Section 2.1 a higher regularity linear problem that we will use in Section 2.2 to obtain a solution to the nonlinear problem (that is, a solution as in Definition 2.1).

### 2.1. The linear problem.

**Proposition 2.3.** Let  $\nu > 0$ . For a fixed  $T > 0$  let  $f \in L^\infty(0, T; L^1 \cap L^\infty) \cap C(0, T; L^2)$  and let  $\mathbf{v}_f = \sigma_1 \nabla \Phi * f$ . Then there exists a unique weak solution  $\rho^\nu \in C(0, T; L^1 \cap L^\infty) \cap L^2(0, T; H^1)$  to

$$\begin{cases} \partial_t \rho^\nu + \mathbf{v}_f \cdot \nabla \rho^\nu = \sigma_2 f \rho^\nu + \nu \Delta \rho^\nu, \\ \rho^\nu(0) = f(0). \end{cases} \quad (2.4)$$

Moreover, if  $f$  also lies in  $L^\infty(0, T; C^\infty)$  then  $\rho^\nu$  also lies in  $L^\infty(0, T; C^\infty)$  and is unique.

*Proof.* We can write (2.4) in weak form as

$$(\partial_t \rho^\nu, \varphi) + a(\rho^\nu, \varphi) = 0$$

a.e. in time for all  $\varphi \in H^1(\mathbb{R}^d)$ , where the bilinear form,  $a: H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \rightarrow \mathbb{R}$ , is given by

$$a(g, \varphi) := \langle \mathbf{v}_f \cdot \nabla g - \sigma_2 f g, \varphi \rangle + \nu \langle \nabla g, \nabla \varphi \rangle.$$

Observe that

$$|a(g, \varphi)| \leq C(f) \|g\|_{H^1(\mathbb{R}^d)} \|\varphi\|_{H^1(\mathbb{R}^d)}$$

and

$$\begin{aligned} a(g, g) &= \langle \mathbf{v}_f \cdot \nabla g - \sigma_2 f g, g \rangle + \nu \|\nabla g\|_{L^2(\mathbb{R}^d)}^2 = \nu \|\nabla g\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2} \int_{\mathbb{R}^d} \mathbf{v}_f \cdot \nabla g^2 - \sigma_2 \int_{\mathbb{R}^d} f g^2 \\ &= \nu \|\nabla g\|_{L^2(\mathbb{R}^d)}^2 - \left( \frac{\sigma_1}{2} + \sigma_2 \right) \int_{\mathbb{R}^d} f g^2 \geq \nu \|\nabla g\|_{L^2(\mathbb{R}^d)}^2 - C(f) \|g\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

The existence of a unique weak solution to (2.4) with  $\rho^\nu \in C(0, T; L^1 \cap L^\infty) \cap L^2(0, T; H^1)$  then follows from [16] (see Theorem 10.9 of [5]). That  $f$  also in  $L^\infty(0, T; C^\infty)$  gives  $\rho^\nu$  in  $L^\infty(0, T; C^\infty)$  follows via a standard bootstrap argument.  $\square$

## 2.2. The nonlinear problem.

**Theorem 2.4.** Fix  $T > 0$  with  $T < (|\sigma_2| \|\rho_0\|_{L^\infty})^{-1}$  or  $T < \infty$  if  $\sigma_2 = 0$ . (Note that  $[0, T]$  is within the time of existence for the inviscid problem—see Theorem 5.2). Let  $\nu > 0$  and assume that  $\rho_0 \in L^1 \cap L^\infty$ . Then there exists a weak solution to  $(GAG_\nu)$  as in Definition 2.1 on the time interval  $[0, T]$  with

$$\|\rho^\nu(t)\|_{L^\infty} \leq \frac{\|\rho_0\|_{L^\infty}}{1 - |\sigma_2| \|\rho_0\|_{L^\infty} t}. \quad (2.5)$$

When  $\sigma_2 \neq 0$ , we have

$$\begin{aligned} \|\rho^\nu(t)\|_{L^q} &\leq \|\rho_0\|_{L^q} (1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)^{-|q^{-1}\sigma_1/\sigma_2+1|} \quad \forall q \in [1, \infty), \\ \|\rho^\nu(t)\|^2 + 2\nu \int_0^t \|\nabla \rho^\nu\|^2 &\leq \|\rho_0\|^2 (1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)^{-|\sigma_1/\sigma_2+2|}. \end{aligned} \quad (2.6)$$

When  $\sigma_2 = 0$ , we have

$$\begin{aligned} \|\rho^\nu(t)\|_{L^q} &\leq \|\rho_0\|_{L^q} \exp(|\sigma_1| q^{-1} \|\rho_0\|_{L^\infty} t) \quad \forall q \in [1, \infty), \\ \|\rho^\nu(t)\|^2 + 2\nu \int_0^t \|\nabla \rho^\nu\|^2 &\leq \|\rho_0\|^2 \exp(|\sigma_1| \|\rho_0\|_{L^\infty} t). \end{aligned} \quad (2.7)$$

**Remark 2.5.** Uniqueness of solutions as in Definition 2.1 is addressed in [1] (at least for  $\sigma_1 + \sigma_2 = 0$ ). In 2D, uniqueness can also be obtained using arguments very close to those we give in Sections 6 and 7 for the vanishing viscosity limit.

To prove Theorem 2.4, we will construct a sequence of approximations. We will obtain the necessary bounds on this sequence in Lemma 2.6, then use these bounds in the proof proper of Theorem 2.4.

The sequence of approximations is defined as follows:

$$\begin{aligned} \rho_0(t, x) &= \rho_0(x), \\ \mathbf{v}_n &= \sigma_1 \nabla \Phi * \rho_{n-1}, \\ \partial_t \rho_n + \mathbf{v}_n \cdot \nabla \rho_n &= \sigma_2 \rho_{n-1} \rho_n + \nu \Delta \rho_n, \\ \rho_n(0) &= \rho_0 \end{aligned} \quad (2.8)$$

for  $n = 1, 2, \dots$ . Note that

$$\operatorname{div} \mathbf{v}_n = \sigma_1 \rho_{n-1}.$$

**Lemma 2.6.** Fix  $T > 0$  with  $T < (|\sigma_2| \|\rho_0\|_{L^\infty})^{-1}$  or  $T < \infty$  if  $\sigma_2 = 0$ . Let  $\nu \geq 0$ ,  $n \geq 0$ ,  $t \in [0, T]$ . We have

$$\begin{aligned} \|\rho_n(t)\|_{L^\infty} &\leq \frac{\|\rho_0\|_{L^\infty}}{1 - |\sigma_2| \|\rho_0\|_{L^\infty} t}, \\ (\partial_t \rho_n) &\text{ is bounded in } L^2(0, T; H^{-1}(\mathbb{R}^d)). \end{aligned} \quad (2.9)$$

When  $\sigma_2 \neq 0$ , we have

$$\begin{aligned} \|\rho_n(t)\|_{L^q} &\leq \|\rho_0\|_{L^q} (1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)^{-|q^{-1}\sigma_1/\sigma_2+1|} \quad \forall q \in [1, \infty), \\ \|\rho_n(t)\|^2 + 2\nu \int_0^t \|\nabla \rho_n\|^2 &= \|\rho_0\|^2 (1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)^{-|\sigma_1/\sigma_2+2|}. \end{aligned} \quad (2.10)$$

When  $\sigma_2 = 0$ , we have

$$\begin{aligned} \|\rho_n(t)\|_{L^q} &\leq \|\rho_0\|_{L^q} \exp(|\sigma_1| q^{-1} \|\rho_0\|_{L^\infty} t) \quad \forall q \in [1, \infty), \\ \|\rho_n(t)\|^2 + 2\nu \int_0^t \|\nabla \rho_n\|^2 &= \|\rho_0\|^2 \exp(|\sigma_1| \|\rho_0\|_{L^\infty} t). \end{aligned} \quad (2.11)$$

*Proof.* We will start with the assumption that  $\rho_0 \in L^1 \cap L^\infty \cap C^\infty$ .

Set as an induction hypothesis that  $\rho_n \in C(0, T; L^1 \cap L^\infty \cap C^\infty)$  and that (2.9)<sub>1</sub> holds.

This is certainly true for  $n = 0$ . Assume it is true up to  $n - 1$ . Then by Proposition 2.3, the equation defining  $\rho_n$  in (2.8)<sub>3</sub> has a solution in  $C(0, T; L^1 \cap L^\infty \cap C^\infty)$ .

Assume that  $q$  is a rational number in  $[2, \infty)$  with  $q = m/n$  in lowest terms for  $m$  even. This insures that  $\rho_n^q \geq 0$ . The conclusions we reach for such rational  $q$ 's will hold for all  $q \in [2, \infty)$  by the continuity of Lebesgue norms.

Multiplying (2.8)<sub>3</sub> by  $\rho_n^{q-1}$  and integrating gives

$$\langle \partial_t \rho_n, \rho_n^{q-1} \rangle + \langle \mathbf{v}_n \cdot \nabla \rho_n, \rho_n^{q-1} \rangle = \sigma_2 \langle \rho_{n-1} \rho_n, \rho_n^{q-1} \rangle + \nu \langle \Delta \rho_n, \rho_n^{q-1} \rangle.$$

But,

$$\begin{aligned} \langle \partial_t \rho_n, \rho_n^{q-1} \rangle &= \frac{1}{q} \int_{\mathbb{R}^d} \partial_t \rho_n^q = \frac{1}{q} \frac{d}{dt} \|\rho_n\|_{L^q}^q, \\ \langle \mathbf{v}_n \cdot \nabla \rho_n, \rho_n^{q-1} \rangle &= \frac{1}{q} \int_{\mathbb{R}^d} \mathbf{v}_n \cdot \nabla \rho_n^q = -\frac{1}{q} \int_{\mathbb{R}^d} \operatorname{div} \mathbf{v}_n \rho_n^q = -\frac{\sigma_1}{q} \int_{\mathbb{R}^d} \rho_{n-1} \rho_n^q, \\ \sigma_2 \langle \rho_{n-1} \rho_n, \rho_n^{q-1} \rangle &= \sigma_2 \int_{\mathbb{R}^d} \rho_{n-1} \rho_n^q, \\ \nu \langle \Delta \rho_n, \rho_n^{q-1} \rangle &= -\nu \langle \nabla \rho_n, \nabla \rho_n^{q-1} \rangle = -(q-1)\nu \int_{\mathbb{R}^d} \rho_n^{q-2} |\nabla \rho_n|^2 \leq 0. \end{aligned}$$

Thus,

$$\frac{d}{dt} \|\rho_n\|_{L^q}^q \leq (\sigma_1 + q\sigma_2) \int_{\mathbb{R}^d} \rho_{n-1} \rho_n^q \leq |\sigma_1 + q\sigma_2| \|\rho_{n-1}\|_{L^\infty} \|\rho_n\|_{L^q}^q. \quad (2.12)$$

Now assume that  $q \in [1, 2)$ , with the same assumption on its rationality as before, and observe that the above argument fails since  $\rho_n^{q-2}$  is singular at  $\rho_n = 0$ . We therefore modify the argument as follows. Fix  $\varepsilon > 0$  and define  $\lambda \in C^\infty(\mathbb{R})$  so that

$$\lambda(x) = \begin{cases} \varepsilon, & 0 \leq x \leq \varepsilon, \\ x, & x \geq 2\varepsilon \end{cases}$$

and so  $\lambda', \lambda'' \geq 0$ . Multiplying (2.8)<sub>3</sub> by  $\lambda'(\rho_n^q) \rho_n^{q-1}$  and integrating gives

$$\langle \partial_t \rho_n, \lambda'(\rho_n^q) \rho_n^{q-1} \rangle + \langle \mathbf{v}_n \cdot \nabla \rho_n, \lambda'(\rho_n^q) \rho_n^{q-1} \rangle = \sigma_2 \langle \rho_{n-1} \rho_n, \lambda'(\rho_n^q) \rho_n^{q-1} \rangle + \nu \langle \Delta \rho_n, \lambda'(\rho_n^q) \rho_n^{q-1} \rangle.$$

But,

$$\begin{aligned}
\langle \partial_t \rho_n, \lambda'(\rho_n^q) \rho_n^{q-1} \rangle &= \frac{1}{q} \int_{\mathbb{R}^d} \partial_t \lambda(\rho_n^q) = \frac{1}{q} \frac{d}{dt} \int_{\mathbb{R}^d} \lambda(\rho_n^q), \\
\langle \mathbf{v}_n \cdot \nabla \rho_n, \lambda'(\rho_n^q) \rho_n^{q-1} \rangle &= \frac{1}{q} \int_{\mathbb{R}^d} \mathbf{v}_n \cdot \nabla \lambda(\rho_n^q) = -\frac{1}{q} \int_{\mathbb{R}^d} \operatorname{div} \mathbf{v}_n \lambda(\rho_n^q) = -\frac{\sigma_1}{q} \int_{\mathbb{R}^d} \rho_{n-1} \lambda(\rho_n^q), \\
\sigma_2 \langle \rho_{n-1} \rho_n, \lambda'(\rho_n^q) \rho_n^{q-1} \rangle &= \sigma_2 \int_{\mathbb{R}^d} \rho_{n-1} \lambda'(\rho_n^q) \rho_n^q, \\
\nu \langle \Delta \rho_n, \lambda'(\rho_n^q) \rho_n^{q-1} \rangle &= -\nu \langle \nabla \rho_n, \nabla (\lambda'(\rho_n^q) \rho_n^{q-1}) \rangle \\
&= -(q-1)\nu \int_{\mathbb{R}^d} \lambda'(\rho_n^q) \rho_n^{q-2} |\nabla \rho_n|^2 - q\nu \int_{\mathbb{R}^d} \lambda''(\rho_n^q) \rho_n^{2(q-1)} |\nabla \rho_n|^2 \leq 0.
\end{aligned}$$

Taking the limit as  $\varepsilon \rightarrow 0$ , we recover the same bound as in (2.12). We use here that  $\lambda' = 0$  in a neighborhood of the origin so that the singularity in  $\rho_n^{q-2}$  is removed.

Let  $t \in [0, T]$ . Applying Gronwall's lemma gives

$$\|\rho_n(t)\|_{L^q}^q \leq \|\rho_0\|_{L^q}^q \exp \left( |\sigma_1 + q\sigma_2| \int_0^t \|\rho_{n-1}(s)\|_{L^\infty} ds \right)$$

so that

$$\|\rho_n(t)\|_{L^q} \leq \|\rho_0\|_{L^q} \exp \left( |q^{-1}\sigma_1 + \sigma_2| \int_0^t \|\rho_{n-1}(s)\|_{L^\infty} ds \right). \quad (2.13)$$

Now, by the induction hypothesis,

$$\begin{aligned}
\int_0^t \|\rho_{n-1}(s)\|_{L^\infty} ds &\leq \int_0^t \frac{\|\rho_0\|_{L^\infty}}{1 - |\sigma_2| \|\rho_0\|_{L^\infty} s} ds \\
&= \begin{cases} -|\sigma_2|^{-1} \log(1 - |\sigma_2| \|\rho_0\|_{L^\infty} t), & \sigma_2 \neq 0, \\ \|\rho_0\|_{L^\infty} t, & \sigma_2 = 0. \end{cases}
\end{aligned}$$

Taking the limit as  $q \rightarrow \infty$  of both sides of (2.13), it follows by the continuity of Lebesgue norms that for  $\sigma_2 \neq 0$ ,

$$\|\rho_n(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} \exp(-\log(1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)) = \|\rho_0\|_{L^\infty} (1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)^{-1}$$

and  $\|\rho_n(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}$  if  $\sigma_2 = 0$ . This shows that the induction hypothesis, (2.9)<sub>1</sub>, holds for  $n$ , and so for all  $n$  by induction.

Returning to (2.13), we see that (2.10)<sub>1</sub> and (2.11)<sub>1</sub> hold.

The bounds in (2.10)<sub>2</sub>, (2.11)<sub>2</sub> follow by not discarding for  $q = 2$  the term above that we observed was never positive. Using  $\partial_t \rho_n = -\mathbf{v}_n \cdot \nabla \rho_n + \nu \Delta \rho_n$ , (2.9)<sub>2</sub> then follows from (2.10)<sub>2</sub> or (2.11)<sub>2</sub> along with Lemma 2.7.

Because the bounds obtained depend only upon the  $L^q$  norms of  $\rho_0$ , we see by the density of  $L^1 \cap C^\infty$  in  $L^1 \cap L^\infty$  that the result holds for  $\rho_0$  in  $L^1 \cap L^\infty$ .  $\square$

**Proof of Theorem 2.4.** Because of the bounds in Lemma 2.6, we can make the same argument for existence of solutions to  $(GAG_\nu)$  as is made for the Navier-Stokes equations on pages 72-73 of [9]. That is, except that  $H^1(\mathbb{R}^d)$  is not compactly embedded in  $L^2(\mathbb{R}^d)$  because  $\mathbb{R}^d$  is an unbounded domain. (The embedding of  $L^2(\mathbb{R}^2)$  into  $H^{-1}(\mathbb{R}^d)$  is, however, continuous, and no compactness is needed for this embedding.) We handle this lack of compactness, however, as Temam does in Remark III.3.2 of [22].

We note that because  $\partial_t \rho^\nu \in L_{loc}^2([0, \infty); H^{-1})$  and not just in  $L_{loc}^1([0, \infty); H^{-1})$ ,  $\rho^\nu$  is equal (a.e.) to a function continuous in  $L^2$ . This is what happens for the Navier-Stokes equations

in 2D versus higher dimension, and is treated in the same manner. (See, for instance, the argument following (3.60) Chapter III of [22].)

The estimates stated in Theorem 2.4 then follow from taking the limit as  $n \rightarrow \infty$  of the bounds obtained in Lemma 2.6.  $\square$

We used Lemma 2.7 above in the proof of Lemma 2.6 and will use it again in later sections.

**Lemma 2.7.** *Let  $\mathbf{w} = \nabla \Phi * f$ . Then*

$$\|\mathbf{w}\|_{LL} \leq C \|f\|_{L^1 \cap L^\infty}.$$

*Proof.* In 2D, this is Lemma 8.1 of [17]. It can be proved in all dimensions in a manner very similar to that of Theorem 3.1 of [23], so we suppress the proof.  $\square$

### 3. SPATIAL DECAY OF VISCOUS SOLUTIONS

Let  $a_R$  be as in Definition 1.1 and let  $b_R(r) = 1 - a_R(x)$ , where  $r := |x|$ .

**Proposition 3.1.** *Let  $\nu \geq 0$ . For any  $R > 0$ , for solutions to  $(GAG_\nu)$  satisfying Definition (2.1),*

$$\|b_R \rho^\nu(t)\|^2 + \nu \int_0^t \|b_R \nabla \rho^\nu\|^2 \leq \left( \|b_R \rho_0^\nu\|^2 + C_0(t) t (\nu + 1) R^{-2} \right) e^{C_0(t)t}. \quad (3.1)$$

Now assume that for some integer  $N \geq 2$  we have  $\|b_R \rho_0^\nu\|^2 \leq C R^{-N}$  for all sufficiently large  $R$ . Then for all sufficiently large  $R$ , we have

$$\|b_R \rho^\nu(t)\|^2 + \nu \int_0^t \|b_R \nabla \rho^\nu\|^2 \leq C_0(t) t (\nu + 1)^2 R^{-N} e^{C_0(t)t}. \quad (3.2)$$

*Proof.* We drop the superscript  $\nu$  for convenience.

Multiplying  $(GAG_\nu)$  by  $b_R^2 \rho$  and integrating, we have

$$\langle \partial_t \rho, b_R^2 \rho \rangle = -\langle \mathbf{v} \cdot \nabla \rho, b_R^2 \rho \rangle + \sigma_2 \langle \rho^2, b_R^2 \rho \rangle + \nu \langle \Delta \rho, b_R^2 \rho \rangle.$$

Now,

$$\begin{aligned} \langle \partial_t \rho, b_R^2 \rho \rangle &= \frac{1}{2} \frac{d}{dt} \|b_R \rho\|^2, \\ -\langle \mathbf{v} \cdot \nabla \rho, b_R^2 \rho \rangle &= -\frac{1}{2} \langle \mathbf{v}, \nabla (b_R^2 \rho^2) \rangle + \frac{1}{2} \langle \mathbf{v}, \rho^2 \nabla b_R^2 \rangle \\ &= \frac{1}{2} \langle \operatorname{div} \mathbf{v}, b_R^2 \rho^2 \rangle + \langle \mathbf{v}, \rho^2 b_R \nabla b_R \rangle = \frac{\sigma_1}{2} \langle \rho, b_R^2 \rho^2 \rangle + \langle \mathbf{v}, \rho^2 b_R \nabla b_R \rangle, \\ \nu \langle \Delta \rho, b_R^2 \rho \rangle &= -\nu \langle \nabla \rho, \nabla (b_R^2 \rho) \rangle = -\nu \langle \nabla \rho, b_R^2 \nabla \rho + \rho \nabla (b_R)^2 \rangle \\ &= -\nu \|b_R \nabla \rho\|^2 - 2\nu \langle b_R \nabla \rho, \rho \nabla b_R \rangle. \end{aligned}$$

We estimate the various terms, taking advantage of Theorems 2.4 and 5.2:

$$\begin{aligned} |\langle \rho, b_R^2 \rho^2 \rangle| &\leq \|\rho\|_{L^\infty} \|b_R^2 \rho^2\|_{L^1} \leq C_0(t) \|b_R \rho\|^2, \\ |\langle \mathbf{v}, \rho^2 b_R \nabla b_R \rangle| &\leq \|\nabla b_R\|_{L^\infty} \|\mathbf{v}\|_{L^\infty} \|\rho\| \|b_R \rho\| \leq C_0(t) (R^{-2} + \|b_R \rho\|^2), \\ |2\nu \langle b_R \nabla \rho, \rho \nabla b_R \rangle| &\leq \frac{\nu}{2} \|b_R \nabla \rho\|^2 + 2\nu \|\rho \nabla b_R\|^2, \\ \nu \|\rho \nabla b_R\|^2 &\leq \nu C_0(t) R^{-2}, \\ \sigma_2 \langle \rho^2, b_R^2 \rho \rangle &\leq |\sigma_2| \|\rho\|_{L^\infty} \|b_R \rho\|^2 \leq C_0(t) \|b_R \rho\|^2. \end{aligned}$$



Combining the estimates above, we have

$$\frac{1}{2} \frac{d}{dt} \|b_R \rho\|^2 + \nu \|b_R \nabla \rho\|^2 \leq C_0(t)(\nu + 1)R^{-2} + \frac{\nu}{2} \|b_R \nabla \rho\|^2 + C_0(t) \|b_R \rho\|^2,$$

or,

$$\frac{d}{dt} \|b_R \rho\|^2 + \nu \|b_R \nabla \rho\|^2 \leq C_0(t)(\nu + 1)R^{-2} + C_0(t) \|b_R \rho\|^2.$$

Applying Gronwall's lemma (using that  $C_0(t)$  increases in  $t$ ) gives (3.1).

Now assume that for some integer  $N \geq 2$  we have  $\|b_R \rho_0\|^2 \leq CR^{-N}$  for all sufficiently large  $R$ . We refine some of our estimates using (3.1). We have,

$$\begin{aligned} |\langle \mathbf{v}, \rho^2 b_R \nabla b_R \rangle| &\leq \|\mathbf{v}\|_{L^\infty} \|\nabla b_R\|_{L^\infty} \|b_R \rho\| \|\rho\|_{L^2(\text{supp } \nabla b_R)} \\ &\leq C_0(t)R^{-1} \left( \|b_R \rho_0\|^2 + C_0(t)t(\nu + 1)R^{-2} \right)^{1/2} e^{C_0(t)t} \|\rho\|_{L^2(\text{supp } \nabla b_R)}, \\ &\leq C_0(t)R^{-1} (CR^{-N} + C_0(t)t(\nu + 1)R^{-2})^{1/2} e^{C_0(t)t} \|\rho\|_{L^2(\text{supp } \nabla b_R)} \end{aligned}$$

and

$$\nu \|\rho \nabla b_R\|^2 \leq \nu \|\nabla b_R\|_{L^\infty}^2 \|\rho\|_{L^2(\text{supp } \nabla b_R)}^2 \leq \nu R^{-2} \|\rho\|_{L^2(\text{supp } \nabla b_R)}^2.$$

But

$$\begin{aligned} \|\rho\|_{L^2(\text{supp } \nabla b_R)} &\leq \|b_{R/2} \rho\| \leq \left( \|b_{R/2} \rho_0\|^2 + C_0(t)t(\nu + 1)R^{-2} \right)^{1/2} e^{C_0(t)t} \\ &\leq (CR^{-N} + C_0(t)t(\nu + 1)R^{-2})^{1/2} e^{C_0(t)t} \end{aligned}$$

follows from (3.1) applied with  $R/2$  in place of  $R$ .

We conclude that

$$\frac{d}{dt} \|b_R \rho\|^2 + \nu \|b_R \nabla \rho\|^2 \leq C_0(t)(\nu + 1)^2(R^{-N-1} + R^{-3}) + C_0(t) \|b_R \rho\|^2$$

holds for all sufficiently large  $R$ . Integrating in time and applying Gronwall's inequality shows, in particular, that (3.2) holds for  $N \leq 3$ . Applying the above process inductively gives (3.2) for all  $N \geq 2$ .  $\square$

**Corollary 3.2.** Fix  $\alpha \geq 0$  and suppose that for some integer  $N > d + 2\alpha + 1$  and some  $R_0 > 0$ , we have  $\|b_R \rho_0^\nu\|^2 \leq CR^{-N}$  for all  $R \geq R_0$ . Then  $|x|^\alpha \rho^\nu(t, x) \in L_x^1(\mathbb{R}^d)$  up to the time of existence.

*Proof.* For convenience, we set  $\rho = \rho^\nu$ . Then

$$\begin{aligned} \| |x|^\alpha \rho \|_{L_x^1(B_{R_0}^C)} &\leq \sum_{k=1}^{\infty} \| |x|^\alpha \rho \|_{L^1(B_{(k+1)R_0} \setminus B_{kR_0})} \leq CR_0^{\alpha + \frac{d}{2}} \sum_{k=1}^{\infty} (k+1)^{\alpha + \frac{d-1}{2}} \|\rho\|_{L^2(B_{(k+1)R_0} \setminus B_{kR_0})} \\ &\leq CR_0^{\alpha + \frac{d}{2}} \sum_{k=1}^{\infty} k^{\alpha + \frac{d-1}{2}} \|b_{\frac{kR_0}{2}} \rho\| \leq C(t, \nu, \alpha, R_0) \sum_{k=1}^{\infty} k^{\alpha + \frac{d-1}{2}} \left( \frac{R_0 k}{2} \right)^{-\frac{N}{2}} \\ &\leq C(t, \nu, \alpha, R_0) \sum_{k=1}^{\infty} k^{\frac{2\alpha + d - 1 - N}{2}} < \infty. \end{aligned}$$

$\square$

**Theorem 3.3.** *Assume that for all  $R \geq R_0$ ,  $\|b_R \rho_0^\nu\|^2 \leq CR^{-N}$  for some  $N > d + 1$ . Then up to the time of existence,*

$$m(\rho^\nu) = m(\rho_0) + (\sigma_1 + \sigma_2) \int_0^t \|\rho^\nu(s)\|^2 ds. \quad (3.3)$$

*Proof.* By Corollary 3.2,  $\rho^\nu(t, x) \in L_x^1(\mathbb{R}^d)$  up to the time of existence. We integrate  $(GAG_\nu)$  over  $\mathbb{R}^d$  to obtain

$$\int_{\mathbb{R}^d} \partial_t \rho^\nu + \int_{\mathbb{R}^d} \mathbf{v}^\nu \cdot \nabla \rho^\nu = \sigma_2 \int_{\mathbb{R}^d} (\rho^\nu)^2 + \nu \int_{\mathbb{R}^d} \Delta \rho^\nu.$$

The bound in (3.1) is sufficient to integrate one term by parts, giving

$$\int_{\mathbb{R}^d} \mathbf{v}^\nu \cdot \nabla \rho^\nu = - \int_{\mathbb{R}^d} \operatorname{div} \mathbf{v}^\nu \rho^\nu = -\sigma_1 \int_{\mathbb{R}^d} (\rho^\nu)^2.$$

For the other two terms we have, formally,

$$\int_{\mathbb{R}^d} \partial_t \rho^\nu = \frac{d}{dt} \int_{\mathbb{R}^d} \rho^\nu, \quad \int_{\mathbb{R}^d} \Delta \rho^\nu = 0. \quad (3.4)$$

Integrating in time then yields (3.3).

Our weak solutions, however, lack the time regularity to obtain the first equality in (3.4), and the spatial regularity and decay to obtain the second. To justify these equalities, we could mollify the initial data and employ a sequence of approximate solutions. Alternately, we could obtain the equivalent of Proposition 3.1 and Corollary 3.2 for the sequence,  $(\rho_n)$ , of approximate solutions employed in the proof of Theorem 2.4. This would lead to the identity,

$$m(\rho_n) = m(\rho_0) + (\sigma_1 + \sigma_2) \int_0^t \langle \rho_{n-1}(s), \rho_n(s) \rangle ds,$$

which in turn yields (3.3) in the limit as  $n \rightarrow \infty$ .  $\square$

When  $\sigma_1 = -\sigma_2$ , as happens for  $(AG_\nu)$ , total mass is conserved, as we can see from (3.3).

#### 4. TOTAL MASS AND INFINITE ENERGY

In dimensions three and higher,  $\rho^\nu \in L^1 \cap L^\infty$  is enough to guarantee membership of  $\mathbf{v}^\nu$  to  $L^2(\mathbb{R}^d)$ . In 2D, however, this is no longer true: the viscous (and inviscid) velocity in 2D will generically have infinite energy, even if it has finite energy at time zero (see, for example, Proposition 3.1.1 of [7]). When dealing only with existence of solutions to  $(GAG_\nu)$ , the infinite energy of 2D velocities is a minor issue. We will need to face this issue directly, however, in Section 6 when we take the vanishing viscosity limit.

In recovering the velocity from its divergence, the total mass (see (2.3)) of the density plays an important, if so far hidden, role in 2D: in short, if the total mass of the density is zero and has sufficient spatial decay, then the velocity will lie in  $L^2$ . We prove this, along with other useful bounds on the velocity, in Lemma 4.3.

Before giving the proof of Lemma 4.3, we must first define the Littlewood-Paley operators. It is classical that there exists two functions  $\chi, \phi \in S(\mathbb{R}^d)$  with  $\operatorname{supp} \chi \subset \{\xi \in \mathbb{R}^d : |\xi| \leq \frac{5}{6}\}$  and  $\operatorname{supp} \hat{\phi} \subset \{\xi \in \mathbb{R}^d : \frac{2}{3} \leq |\xi| \leq \frac{5}{3}\}$ , such that, if for every  $j \geq 0$  we set  $\phi_j(x) = 2^{jd} \phi(2^j x)$ , then

$$\hat{\chi} + \sum_{j \geq 0} \hat{\phi}_j = \hat{\chi} + \sum_{j \geq 0} \hat{\phi}(2^{-j} \cdot) \equiv 1.$$

For  $f \in S'(\mathbb{R}^d)$  and  $j \geq -1$ , define the Littlewood-Paley operators  $\Delta_j$  by

$$\Delta_j f = \begin{cases} \chi * f, & j = -1 \\ \phi_j * f, & j \geq 0. \end{cases}$$

We make use of the following lemma throughout the paper. A proof of the lemma can be found in [6], chapter 2. Below,  $C_{a,b}(0)$  denotes the annulus with inner radius  $a$  and outer radius  $b$ .

**Lemma 4.1.** (*Bernstein's Lemma*) *Let  $r_1$  and  $r_2$  satisfy  $0 < r_1 < r_2 < \infty$ , and let  $p$  and  $q$  satisfy  $1 \leq p \leq q \leq \infty$ . There exists a positive constant  $C$  such that for every integer  $k$ , if  $u$  belongs to  $L^p(\mathbb{R}^d)$ , and  $\text{supp } \hat{u} \subset B_{r_1\lambda}(0)$ , then*

$$\sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^k \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}. \quad (4.1)$$

Furthermore, if  $\text{supp } \hat{u} \subset C_{r_1\lambda, r_2\lambda}(0)$ , then

$$C^{-k} \lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^k \lambda^k \|u\|_{L^p}. \quad (4.2)$$

The following Littlewood-Paley definition of Holder spaces will be useful. This definition is equivalent to the classical definition of Holder spaces when  $\alpha$  is a positive non-integer (see, for example, [6], chapter 2).

**Definition 4.2.** *For  $\alpha \in \mathbb{R}$ , the space  $C_*^\alpha$  is the set of functions  $f$  such that*

$$\sup_{j \geq -1} 2^{j\alpha} \|\Delta_j f\|_{L^\infty} < \infty.$$

We set

$$\|f\|_{C_*^\alpha} = \sup_{j \geq -1} 2^{j\alpha} \|\Delta_j f\|_{L^\infty}.$$

When  $\alpha$  is a positive non-integer, we will often write  $C^\alpha$  in place of  $C_*^\alpha$ , in view of the equivalence between the two spaces.

**Lemma 4.3.** *Let  $\rho \in L^1 \cap L^\infty(\mathbb{R}^d)$ . For all  $p \in (d/(d-1), \infty]$ ,  $d \geq 2$ ,*

$$\|\nabla \Phi * \rho\|_{L^p} \leq C \|\rho\|_{L^1 \cap L^p}. \quad (4.3)$$

If  $d \geq 3$  then for all  $p \in (d/(d-2), \infty]$ ,

$$\|\Phi * \rho\|_{L^p} \leq C(p) \|\rho\|_{L^1 \cap L^p}, \quad \|\nabla \Phi * \rho\| \leq C \|\rho\|_{L^1 \cap L^\infty}^{\frac{1}{2}} \|\rho\|_{L^1}^{\frac{1}{2}}. \quad (4.4)$$

Moreover, for  $d \geq 3$ , let  $p_1, p_2 \in \mathbb{R}$  with  $1 \leq p_1 < d/2 < p_2 \leq \infty$ . Then

$$\|\Phi * \rho\|_{L^\infty} \leq C \|\rho\|_{L^{p_1} \cap L^{p_2}}, \quad \|\nabla \Phi * \rho\| \leq C \|\rho\|_{L^{p_1} \cap L^{p_2}}^{\frac{1}{2}} \|\rho\|_{L^1}^{\frac{1}{2}}. \quad (4.5)$$

Let  $d = 2$  with  $m(\rho) = 0$  and  $|x|^\varepsilon \rho(x) \in L_x^1(\mathbb{R}^2)$  for some  $\varepsilon \in (0, 1]$ . For all  $p \in (\frac{2}{\varepsilon}, \infty]$ ,  $\Phi * \rho \in L^p$  and

$$\|\Phi * \rho\|_{L^p} \leq C(\| |x|^\varepsilon \rho(x) \|_{L_x^1} + \|\rho\|_{L^1 \cap L^\infty}), \quad \|\nabla \Phi * \rho\|^2 \leq C(\| |x|^\varepsilon \rho(x) \|_{L_x^1} + \|\rho\|_{L^1 \cap L^\infty}) \|\rho\|_{L^1}. \quad (4.6)$$

*Proof.* Let  $a$  be as in Definition 1.1. First observe that

$$\|\nabla \Phi * \rho\|_{L^p} \leq \|a \nabla \Phi\|_{L^1} \|\rho\|_{L^p} + \|(1-a) \nabla \Phi\|_{L^p} \|\rho\|_{L^1} < \infty$$

for all  $p > d/(d-1)$ , giving (4.3). For  $d \geq 3$ ,

$$\|\Phi * \rho\|_{L^p} \leq \|a \Phi\|_{L^1} \|\rho\|_{L^p} + \|(1-a) \Phi\|_{L^p} \|\rho\|_{L^1} < \infty$$

for all  $p > d/(d-2)$ . In particular,  $\|\Phi * \rho\|_{L^\infty} \leq C \|\rho\|_{L^1 \cap L^\infty}$ . Hence, for  $d \geq 3$ , we can apply Lemma 4.5, which gives

$$\|\nabla \Phi * \rho\|^2 = - \int_{\mathbb{R}^d} (\Phi * \rho)(\Delta \Phi * \rho) \leq \|\Phi * \rho\|_{L^\infty} \|\Delta \Phi * \rho\|_{L^1} = \|\Phi * \rho\|_{L^\infty} \|\rho\|_{L^1},$$

which leads to (4.4).

For the remainder of the proof, let  $\mathcal{F}$  denote the Fourier transform operator.

To establish the lemma for  $d = 2$ , we will first show that since  $|x|^\varepsilon \rho(x)$  belongs to  $L_x^1(\mathbb{R}^2)$ ,  $\hat{\rho}$  belongs to  $C^\varepsilon(\mathbb{R}^2)$ . To see this, note that for  $j \geq 0$ ,

$$\begin{aligned} \|\Delta_j \hat{\rho}\|_{L^\infty} &= \|\phi_j * \hat{\rho}\|_{L^\infty} \leq \|\mathcal{F}(\phi_j * \hat{\rho})\|_{L^1} = \|\hat{\phi}_j(x) \rho(-x)\|_{L_x^1} = \|\hat{\phi}_j(x) |x|^{-\varepsilon} |x|^\varepsilon \rho(-x)\|_{L_x^1} \\ &\leq \|\hat{\phi}_j(x) |x|^{-\varepsilon}\|_{L_x^\infty(\text{supp } \hat{\phi}_j)} \| |x|^\varepsilon \rho(x) \|_{L_x^1} \leq C 2^{-j\varepsilon} \| |x|^\varepsilon \rho(x) \|_{L_x^1}. \end{aligned}$$

Also note that, by Bernstein's Lemma (or Young's convolution inequality),  $\|\Delta_{-1} \hat{\rho}\|_{L^\infty} \leq C \|\hat{\rho}\|_{L^2} = C \|\rho\|_{L^2}$ . We conclude that

$$\begin{aligned} \|\hat{\rho}\|_{C^\varepsilon} &= \sup_{j \geq -1} 2^{j\varepsilon} \|\Delta_j \hat{\rho}\|_{L^\infty} \leq C \|\rho\|_{L^2} + \sup_{j \geq 0} 2^{j\varepsilon} \|\Delta_j \hat{\rho}\|_{L^\infty} \\ &\leq C(\|\rho\|_{L^2} + \| |x|^\varepsilon \rho(x) \|_{L_x^1}). \end{aligned}$$

Since  $\hat{\rho}(0) = 0$ , we can write

$$\begin{aligned} |\hat{\rho}(\xi)| &= |\hat{\rho}(\xi) - \hat{\rho}(0)| \leq C(\|\rho\|_{L^2} + \| |x|^\varepsilon \rho(x) \|_{L_x^1}) |\xi - 0|^\varepsilon \\ &= C(\|\rho\|_{L^2} + \| |x|^\varepsilon \rho(x) \|_{L_x^1}) |\xi|^\varepsilon \end{aligned} \quad (4.7)$$

for each  $\xi \in \mathbb{R}^2$ . The estimate (4.7) implies that  $\mathcal{F}(\Delta_{-1}(\Phi * \rho))$  belongs to  $L^r(\mathbb{R}^2)$  for all  $r < \frac{2}{2-\varepsilon}$ . Thus, using the Hausdorff-Young inequality, for any  $p$  satisfying  $1/p + 1/q = 1$ , with  $q < \frac{2}{2-\varepsilon}$ , we can write

$$\begin{aligned} \|\Phi * \rho\|_{L^p} &\leq \|\Delta_{-1}(\Phi * \rho)\|_{L^p} + \sum_{j \geq 0} \|\Delta_j(\Phi * \rho)\|_{L^p} \\ &\leq \|\mathcal{F}(\Delta_{-1}(\Phi * \rho))\|_{L^q} + \sum_{j \geq 0} 2^{-2j} \|\Delta_j(\partial_k \partial_l \Phi * \rho)\|_{L^p} \\ &\leq \|\mathcal{F}(\Delta_{-1}(\Phi * \rho))\|_{L^q} + C \|\rho\|_{L^p} \leq C(\| |x|^\varepsilon \rho(x) \|_{L_x^1} + \|\rho\|_{L^1 \cap L^\infty}), \end{aligned}$$

where we also used Bernstein's Lemma to get the second inequality. To obtain the third inequality, for  $p < \infty$  we used boundedness of the Riesz transforms on  $L^p$ , and for  $p = \infty$  we used a classical lemma (for a proof, see Lemma 4.2 of [10]).

We conclude, in particular, that when  $d = 2$ ,  $\|\Phi * \rho\|_{L^\infty} \leq C(\| |x|^\varepsilon \rho(x) \|_{L_x^1} + \|\rho\|_{L^1 \cap L^\infty})$ . Since  $\Phi * \rho$  belongs to  $L^{p_1}$  for some  $p_1 < \infty$ , and  $\nabla \Phi * \rho$  belongs to  $L^{p_2}$  for all  $p_2 > 2$  by (4.3), we can apply Lemma 4.5, which gives

$$\|\nabla \Phi * \rho\|^2 \leq \|\Phi * \rho\|_{L^\infty} \|\rho\|_{L^1},$$

which leads to (4.6).

For (4.5) for  $d \geq 3$ , instead of the bound  $\|\Phi * \rho\|_{L^\infty} \leq C \|\rho\|_{L^1 \cap L^\infty}$ , we use the bound

$$\|\Phi * \rho\|_{L^\infty} \leq \|a\Phi\|_{L^{p'_2}} \|\rho\|_{L^{p_2}} + \|(1-a)\Phi\|_{L^{p'_1}} \|\rho\|_{L^{p_1}},$$

where the primes represent Hölder conjugates. This is finite since  $p'_2 < d/(d-2)$  and  $\Phi \in L_{loc}^{p'_2}(\mathbb{R}^d)$ , while  $p'_1 > d/(d-2)$  and  $\Phi$  decays like an  $L^{p'_1}$ -function.  $\square$

To treat densities in  $\mathbb{R}^2$  having nonzero total mass, as we will need to do in Section 7, we will subtract from the associated velocity field a radially symmetric velocity field,  $\tau_0$ . We do this in analogy with the definition of the stationary solution to the Euler equations used to obtain the radial-energy decomposition of a 2D velocity field in [7, 17].

**Definition 4.4.** Fix a radially symmetric function  $g_0 \in C_c^\infty(\mathbb{R}^2)$  having total mass 1. We will abuse notation by writing both  $g_0(x)$  and  $g_0(r)$ , where  $x \in \mathbb{R}^2$  and  $r = |x|$ . Define

$$\tau_0(x) = f(r)x, \quad f(r) := \frac{1}{r^2} \int_0^r \eta g_0(\eta) d\eta.$$

Being a radially directed vector field,  $\tau_0$ , is a gradient. We see that

$$\begin{aligned} \operatorname{div} \tau_0 &= 2f + x^i \partial_i f = 2f + x^i \frac{x^i}{r} \partial_r f = 2f + r \partial_r f \\ &= 2f - 2r \frac{1}{r^3} \int_0^r \eta g_0(\eta) d\eta + r \frac{r g_0(r)}{r^2} = 2f - 2f + g_0(r) = g_0(r). \end{aligned}$$

Hence, also,  $\tau_0 = \nabla \Phi * g_0$ .

We used the following technical lemma in the proof of Lemma 4.3, above.

**Lemma 4.5.** Let  $\varphi \in L^{p_1} \cap L^\infty(\mathbb{R}^d)$  with  $\nabla \varphi \in L^{p_2} \cap L^\infty(\mathbb{R}^d)$  and  $\Delta \varphi \in L^1 \cap L^\infty(\mathbb{R}^d)$ . If  $1/p_1 + 1/p_2 \geq (d-1)/d$  then  $\nabla \varphi \in L^2(\mathbb{R}^d)$ . If  $1/p_1 + 1/p_2 > (d-1)/d$  then

$$\|\nabla \varphi\|^2 = - \int_{\mathbb{R}^d} \varphi \Delta \varphi. \quad (4.8)$$

*Proof.* Let  $a_R$  be as in Definition 1.1. Assume first that  $\varphi$  is also in  $C^\infty(\mathbb{R}^d)$ . Then

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla \varphi|^2 &= \lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} a_R \nabla \varphi \cdot \nabla \varphi = - \lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} \operatorname{div}(a_R \nabla \varphi) \varphi \\ &= - \lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} a_R \Delta \varphi \varphi - \lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} (\nabla a_R \cdot \nabla \varphi) \varphi \\ &= - \int_{\mathbb{R}^d} \Delta \varphi \varphi - \lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} (\nabla a_R \cdot \nabla \varphi) \varphi. \end{aligned}$$

For the first equality, the properties of  $a$  allow us to apply the monotone convergence theorem (we may obtain  $\infty$ , though). The one limit we evaluated is valid because  $a_R \Delta \varphi \rightarrow \Delta \varphi$  in  $L^1(\mathbb{R}^d)$ . For the remaining limit, we have

$$\left| \int_{\mathbb{R}^d} (\nabla a_R \cdot \nabla \varphi) \varphi \right| \leq \|\nabla a_R\|_{L^\infty} \|1\|_{L^p(\operatorname{supp} a_R)} \|\nabla \varphi\|_{L^{p_1}} \|\varphi\|_{L^{p_2}} \leq \frac{C}{R} R^{\frac{d}{p}} = C R^{\frac{d}{p}-1}. \quad (4.9)$$

By assumption,  $1 = \frac{1}{p} + \frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p} + \frac{d-1}{d}$ , so  $p \geq d$ . If  $\frac{1}{p_1} + \frac{1}{p_2} > \frac{d-1}{d}$ , so that  $p > d$ , then we conclude that the remaining limit vanishes, from which (4.8) follows.  $\square$

## 5. THE INVISCID PROBLEM

Well-posedness of weak solutions to  $(AG_0)$  locally in time having bounded, compactly supported density as well as classical solutions having Hölder regularity is proved in [3]. All the solutions constructed were also Lagrangian solutions. The approach in [3] can be adapted to apply to the more general equations in  $(GAG_0)$  for initial density in  $L^1 \cap L^\infty$ , and lead to Theorems 5.2 and 5.3, below. Alternately, the economical and elegant proof of the existence and uniqueness of 2D solutions to the Euler equations given by Marchioro and Pulvirenti in [19], which originates in their earlier text [18], can be adapted to obtain the same results.

In brief, the authors of [3] first construct smooth solutions then use a sequence of approximate smooth solutions to obtain a weak solution by demonstrating convergence of the flow maps (as in [17]). This approach is reversed in [19], where weak (Lagrangian) solutions are first constructed by obtaining the convergence of a sequence of flow maps for approximating linearizations of the 2D Euler equations. A very simple argument then shows that regularity of the initial data is propagated over time. Considerable complications arise when adapting Marchioro and Pulvirenti's to apply to  $(GAG_0)$ , because the underlying velocity field is not divergence-free (analogous complications are dealt with in [3]). This requires the assumption of some regularity on the initial data to obtain weak solutions, an assumption that is only removed a posteriori via a separate (but very similar) iteration to that used to prove existence. Since the focus of this paper is on the vanishing viscosity limit, we do not give the details of this alternate approach here.

Formally, if  $\rho = \rho^0$  solves  $(GAG_0)$  and  $X$  is the flow map for  $\mathbf{v} = \mathbf{v}^0$ , then

$$\frac{d}{dt}\rho(t, X(t, x)) = \sigma_2 \rho(t, X(t, x))^2.$$

Integrating along flow lines gives

$$\rho(t, X(t, x)) = \frac{\rho_0(x)}{1 - \sigma_2 t \rho_0(x)}. \quad (5.1)$$

This motivates the following definition of a Lagrangian solution to  $(GAG_0)$ :

**Definition 5.1.** *Let  $\rho \in L_{loc}^\infty([0, \infty); L^1 \cap L^\infty(\mathbb{R}^d)) \cap C([0, \infty); L^2(\mathbb{R}^d))$  and let  $\mathbf{v} := \sigma_1 \nabla \Phi * \rho$ . By Lemma 2.7,  $\mathbf{v} \in C([0, \infty); LL(\mathbb{R}^d))$ , where  $LL(\mathbb{R}^d)$  is the space of bounded log-Lipschitz vector fields, and so  $\mathbf{v}$  has a unique classical flow map,  $X$ . We say that  $\rho$  is a Lagrangian solution to the inviscid aggregation equations  $(GAG_0)$  with initial density  $\rho_0 \in L^1 \cap L^\infty$  if*

$$\rho(t, x) = \frac{\rho_0(X^{-1}(t, x))}{1 - \sigma_2 t \rho_0(X^{-1}(t, x))}$$

for all  $t \geq 0$ ,  $x \in \mathbb{R}^d$ .

The form of  $\rho$  in (5.1) also yields a sharp time of existence for our Lagrangian solutions. If we do not consider the sign of  $\rho_0$ , we obtain an upper limit on the existence time that is the same as that for viscous solutions in Theorem 2.4. Hence, we should expect that if, say,  $\sigma_2 < 0$  and  $\rho_0 > 0$ , so that the inviscid solution exists for all time, then the existence time for viscous solutions might be considerably longer than the bound given in Theorem 2.4. An open question is whether for all sufficiently small viscosity, viscous solutions to  $(GAG_\nu)$  exist for as long as the inviscid solution exists, as was established for the 3D Navier-Stokes and Euler equations in [8]. (Issues of existence times of viscous solutions in relation to the total mass of  $\rho_0$  have been well-studied: see [21].)

We have the existence of weak and of strong solutions to  $(GAG_0)$ :

**Theorem 5.2.** *Fix  $T > 0$  with  $T < (|\sigma_2| \|\rho_0\|_{L^\infty})^{-1}$  or  $T < \infty$  if  $\sigma_2 = 0$ . Assume that  $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$  is compactly supported. Then there exists a unique weak solution to  $(GAG_0)$  as in Definition 2.1 on the time interval  $[0, T]$ . This weak solution is the unique Lagrangian solution. Moreover, (2.5) holds.*

We establish first that with additional regularity of the initial density, a weak solution is a classical solution.

**Theorem 5.3.** *Assume that  $\rho_0 \in C^{k,\alpha}(\mathbb{R}^d)$  and compactly supported,  $k \geq 0$ ,  $\alpha \in (0, 1)$ . There exists a unique classical solution,  $\rho \in L^\infty(0, T; C^{k,\alpha})$ , to  $(GAG_0)$ . Moreover,*

$$\|\rho(t)\|_{C^{k,\alpha}} \leq C(t, |\sigma_1|, \|\rho_0\|_{L^1}, \|\rho_0\|_{C^{k,\alpha}}). \quad (5.2)$$

## 6. THE VANISHING VISCOSITY LIMIT FOR $(GAG_\nu)$ FOR VELOCITIES IN $L^2$

In this section we consider the vanishing viscosity limit (VV) (see Section 1) for any  $\sigma_1, \sigma_2$  when  $d \geq 3$  and  $\sigma_1 + \sigma_2 = 0$  when  $d = 2$ . In both these cases,  $\mathbf{v}^\nu - \mathbf{v}^0$  remains in  $L^2(\mathbb{R}^d)$ . In Section 7 we consider the general situation in 2D.

For the remainder of this paper, we will assume that the initial density is compactly supported. This gives, through the results of Section 3, rapid spatial decay of the viscous solutions and no difficulties obtaining the identities for the total mass of the density (recall the definition of the total mass in (2.3).) In particular, we have Proposition 6.1.

**Proposition 6.1.** *Assume that  $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$  is compactly supported. Let  $\nu \geq 0$  and let  $\mu = \rho^\nu - \rho^0$ . Then*

$$\begin{aligned} m(\mu(t)) &= (\sigma_1 + \sigma_2) \int_0^t \langle \mu(s), \rho^0(s) + \rho^\nu(s) \rangle ds, \\ |m(\mu(t))| &\leq |\sigma_1 + \sigma_2| \int_0^t (\|\rho^0(s)\| + \|\rho^\nu(s)\|) \|\mu(s)\| ds. \end{aligned}$$

*Proof.* This follows from (3.3). □

Though total mass of  $\rho^\nu - \rho^0$  is zero at time zero, there is no reason to expect, based upon the identity in Proposition 6.1, that  $m(\mu(t))$  remains zero. Proposition 6.1 does show, however, that, as  $\nu \rightarrow 0$ ,  $m(\mu)$  vanishes if  $\|\mu\|$  vanishes. This will be very useful to us in Section 7.

**Theorem 6.2.** *Let  $T$  be as in Theorem 2.4. Assume that  $\rho_0$  is compactly supported and in  $C_C^\alpha$  for some  $\alpha > 1$ . Also assume  $d \geq 3$  or  $\sigma_1 + \sigma_2 = 0$ . Then for all  $\nu \leq 1$  and  $t \in [0, T]$ ,*

$$\|(\mathbf{v}^\nu - \mathbf{v}^0)(t)\|_{H^1}^2 + \|(\rho^\nu - \rho^0)(t)\|^2 + \nu \int_0^t \|(\rho^\nu - \rho^0)(s)\|^2 ds \leq C_0(t) t \nu e^{C_0(t)}.$$

*Proof.* Define

$$\mu = \rho^\nu - \rho^0, \quad \mathbf{w} = \mathbf{v}^\nu - \mathbf{v}^0. \quad (6.1)$$

Then  $\operatorname{div} \mathbf{w} = \sigma_1 \mu$ , and  $\mathbf{w} \in L^2(\mathbb{R}^d)$  for all time by Lemma 4.3, since  $m(\mu) = 0$  by Proposition 6.1.

Taking  $(GAG_\nu) - (GAG_0)$  gives

$$\partial_t \mu + \mathbf{w} \cdot \nabla \rho^0 + \mathbf{v}^\nu \cdot \nabla \mu = \sigma_2 \mu (\rho^0 + \rho^\nu) + \nu \Delta \rho^\nu.$$

Noting that each term above is at least in  $L^2(0, T; H^{-1})$  by Theorems 2.4 and 5.2, we can take the pairing of the above equation with  $\varphi \in C_C^\infty([0, T] \times \mathbb{R}^d)$ , giving

$$(\partial_t \mu, \varphi) = -\langle \mathbf{w} \cdot \nabla \rho^0, \varphi \rangle - \langle \mathbf{v}^\nu \cdot \nabla \mu, \varphi \rangle + \sigma_2 \langle \mu (\rho^0 + \rho^\nu), \varphi \rangle + \nu \langle \Delta \rho^\nu, \varphi \rangle. \quad (6.2)$$

By density, (6.2) holds as well if we set  $\varphi = \mu \in L^2(0, T; H^1)$ . Then,

$$\begin{aligned}
(\partial_t \mu, \varphi) &= \frac{1}{2} \frac{d}{dt} \|\mu\|^2, \\
-\langle \mathbf{v}^\nu \cdot \nabla \mu, \varphi \rangle &= -\frac{1}{2} \langle \mathbf{v}^\nu, \nabla \mu^2 \rangle = \frac{1}{2} \langle \operatorname{div} \mathbf{v}^\nu, \mu^2 \rangle = \frac{\sigma_1}{2} \langle \rho^\nu, \mu^2 \rangle \leq \frac{\sigma_1}{2} \|\rho^\nu\|_{L^\infty} \|\mu\|^2 \\
&\leq C_0(t) \|\mu\|^2, \\
\sigma_2 \langle \mu(\rho^0 + \rho^\nu), \varphi \rangle &\leq |\sigma_2| \|\rho^0 + \rho^\nu\|_{L^\infty} \|\mu\|^2 \leq C_0(t) \|\mu\|^2, \\
\nu \langle \Delta \rho^\nu, \varphi \rangle &= -\nu \langle \nabla \rho^\nu, \nabla \mu \rangle = -\nu \langle \nabla \mu, \nabla \mu \rangle - \nu \langle \nabla \rho^0, \nabla \mu \rangle \\
&\leq -\nu \langle \nabla \mu, \nabla \mu \rangle + \frac{\nu}{2} \|\nabla \rho^0\|^2 + \frac{\nu}{2} \|\nabla \mu\|^2 \leq C_0(t) \nu - \frac{\nu}{2} \|\nabla \mu\|^2.
\end{aligned}$$

For the time derivative, we used Theorem 3 Section 5.9 of [11] (or see [22]).

To estimate the term  $-\langle \mathbf{w} \cdot \nabla \rho^0, \varphi \rangle$ , we consider the cases  $d = 2$  and  $d = 3$  separately. Note that when  $d = 3$ , by the Hardy-Littlewood-Sobolev inequality,  $\|\mathbf{w}\|_{L^6} \leq C\|\mu\|$ . Therefore, when  $d = 3$ ,

$$\begin{aligned}
-\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle &\leq \|\mathbf{w} \cdot \nabla \rho^0\| \|\mu\| \leq \|\mathbf{w}\|_{L^6} \|\nabla \rho^0\|_{L^3} \|\mu\| \\
&\leq C\|\mu\| \|\nabla \rho^0\|_{L^3} \|\mu\| = C\|\mu\|^2 \|\nabla \rho^0\|_{L^3} \leq C_0(t) \|\mu\|^2.
\end{aligned}$$

For the case  $d = 2$ , the Hardy-Littlewood-Sobolev inequality does not yield the desired estimate, but we have

$$-\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle \leq \|\mathbf{w} \cdot \nabla \rho^0\| \|\mu\| \leq \|\mathbf{w}\|_{L^p} \|\nabla \rho^0\|_{L^q} \|\mu\| \quad (6.3)$$

where  $1/p + 1/q = 1/2$ . Now note that for  $p \in (2, \infty)$ , by Bernstein's Lemma and boundedness of Calderon-Zygmund operators on  $L^2$  (recall the definition of  $\Delta_j$  in Section 4),

$$\begin{aligned}
\|\mathbf{w}\|_{L^p} &\leq \|\Delta_{-1} \mathbf{w}\|_{L^p} + \sum_{j \geq 0} \|\Delta_j \mathbf{w}\|_{L^p} \leq C\|\Delta_{-1} \mathbf{w}\| + C \sum_{j \geq 0} 2^{-j} \|\Delta_j \nabla \nabla \Phi * \mu\|_{L^p} \\
&\leq C\|\mathbf{w}\| + C \sum_{j \geq 0} 2^{-j} 2^{2j(1/2-1/p)} \|\Delta_j \nabla \nabla \Phi * \mu\| \\
&\leq C\|\mathbf{w}\| + C \sum_{j \geq 0} 2^{-(2j/p)} \|\mu\| \leq C(\|\mathbf{w}\| + \|\mu\|).
\end{aligned} \quad (6.4)$$

Substituting this estimate into (6.3) gives, for any fixed  $q \in (2, \infty)$ ,

$$-\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle \leq \|\mathbf{w} \cdot \nabla \rho^0\| \|\mu\| \leq C\|\nabla \rho^0\|_{L^q} (\|\mathbf{w}\| + \|\mu\|) \|\mu\| \leq C_0(t) (\|\mathbf{w}\| + \|\mu\|) \|\mu\|. \quad (6.5)$$

Applying the above estimates to (6.2), we see that for  $d = 3$ ,

$$\frac{d}{dt} \|\mu\|^2 + \nu \|\nabla \mu\|^2 \leq C_0(t) \nu + C_0(t) \|\mu\|^2, \quad (6.6)$$

while, for  $d = 2$ ,

$$\frac{d}{dt} \|\mu\|^2 + \nu \|\nabla \mu\|^2 \leq C_0(t) \nu + C_0(t) \|\mathbf{w}\|^2 + C_0(t) \|\mu\|^2. \quad (6.7)$$

After integrating (6.6) in time and applying Gronwall's Lemma, we can conclude that  $\rho_\nu$  converges to  $\rho$  in  $L^\infty([0, T]; L^2(\mathbb{R}^3))$  as  $\nu$  approaches zero. However, we must obtain a bound on  $\|\mathbf{w}\|$  for both  $d = 2$  and  $d = 3$  below in order to obtain the estimate in Theorem 6.2 on the difference of velocities in  $H^1$ . Therefore, in what follows, we utilize (6.7) for both  $d = 2$  and  $d = 3$ .



Integrating (6.7) in time, we have

$$\|\mu(t)\|^2 + \nu \int_0^t \|\nabla \mu(s)\|^2 ds \leq C_0(t)t\nu + \int_0^t C_0(s) \left( \|\mathbf{w}(s)\|^2 + \|\mu(s)\|^2 \right) ds. \quad (6.8)$$

We return to (6.2), sticking for the moment with an unspecified  $\varphi \in L^2(0, T; H^1)$ , but integrating several of the terms by parts in a different manner than above. We have,

$$\begin{aligned} (\partial_t \mu, \varphi) &= \sigma_1^{-1} (\partial_t \operatorname{div} \mathbf{w}, \varphi) = -\sigma_1^{-1} \langle \partial_t \mathbf{w}, \nabla \varphi \rangle, \\ -\langle \mathbf{w} \cdot \nabla \rho^0, \varphi \rangle &= -\langle \varphi \mathbf{w}, \nabla \rho^0 \rangle = \langle \operatorname{div}(\varphi \mathbf{w}), \rho^0 \rangle = \langle \varphi \operatorname{div} \mathbf{w}, \rho^0 \rangle + \langle \nabla \varphi \cdot \mathbf{w}, \rho^0 \rangle \\ &= \sigma_1 \langle \varphi \mu, \rho^0 \rangle + \langle \nabla \varphi \cdot \mathbf{w}, \rho^0 \rangle, \\ -\langle \mathbf{v}^\nu \cdot \nabla \mu, \varphi \rangle &= -\langle \varphi \mathbf{v}^\nu, \nabla \mu \rangle = \langle \operatorname{div}(\varphi \mathbf{v}^\nu), \mu \rangle = \langle \varphi \operatorname{div} \mathbf{v}^\nu, \mu \rangle + \langle \nabla \varphi \cdot \mathbf{v}^\nu, \mu \rangle \\ &= \sigma_1 \langle \varphi \rho^\nu, \mu \rangle + \langle \nabla \varphi \cdot \mathbf{v}^\nu, \mu \rangle, \\ \nu(\Delta \rho^\nu, \varphi) &= -\nu \langle \nabla \rho^\nu, \nabla \varphi \rangle. \end{aligned} \quad (6.9)$$

From (6.2), then, we have

$$\begin{aligned} -\sigma_1^{-1} \langle \partial_t \mathbf{w}, \nabla \varphi \rangle &= \sigma_1 \langle \varphi \mu, \rho^0 \rangle + \langle \nabla \varphi \cdot \mathbf{w}, \rho^0 \rangle + \sigma_1 \langle \varphi \rho^\nu, \mu \rangle + \langle \nabla \varphi \cdot \mathbf{v}^\nu, \mu \rangle \\ &\quad + \sigma_2 \langle \mu(\rho^0 + \rho^\nu), \varphi \rangle - \nu \langle \nabla \rho^\nu, \nabla \varphi \rangle \\ &= \langle \nabla \varphi \cdot \mathbf{w}, \rho^0 \rangle + \langle \nabla \varphi \cdot \mathbf{v}^\nu, \mu \rangle + (\sigma_1 + \sigma_2) \langle \mu(\rho^0 + \rho^\nu), \varphi \rangle - \nu \langle \nabla \rho^\nu, \nabla \varphi \rangle. \end{aligned} \quad (6.10)$$

Now set  $\varphi = \sigma_1 \Phi * \mu$ . For  $d \geq 3$ ,  $\varphi$  lies in  $L^2(0, T; \dot{H}^1)$  by Theorem 2.4 and Lemma 4.3, so (6.10) continues to hold by the density of  $C_c^\infty([0, T] \times \mathbb{R}^d)$  in  $L^2(0, T; \dot{H}^1)$ . The same is true for  $d = 2$  when  $\sigma_1 + \sigma_2 = 0$  (though then the only term on the right-hand side of (6.10) involving  $\varphi$  without a gradient vanishes). Hence, in both cases covered by this theorem, (6.10) holds for  $\varphi = \sigma_1 \Phi * \mu$ . Then  $\nabla \varphi = \mathbf{w}$ , and we find that

$$\begin{aligned} \sigma_1^{-1} \langle \partial_t \mathbf{w}, \mathbf{w} \rangle &= \frac{\sigma_1^{-1}}{2} \frac{d}{dt} \|\mathbf{w}\|^2 \\ &= -\langle |\mathbf{w}|^2, \rho^0 \rangle - \langle \mathbf{w} \cdot \mathbf{v}^\nu, \mu \rangle + \nu \langle \nabla \rho^\nu, \mathbf{w} \rangle - \sigma_1(\sigma_1 + \sigma_2) \langle \mu \eta, \Phi * \mu \rangle, \end{aligned} \quad (6.11)$$

where  $\eta := \rho^0 + \rho^\nu$ . But,

$$\begin{aligned} |\langle |\mathbf{w}|^2, \rho^0 \rangle| &\leq \|\rho^0\|_{L^\infty} \|\mathbf{w}\|^2 \leq C_0(t) \|\mathbf{w}\|^2, \\ |\langle \mathbf{w} \cdot \mathbf{v}^\nu, \mu \rangle| &\leq \|\mathbf{v}^\nu\|_{L^\infty} \|\mathbf{w}\| \|\mu\| \leq C_0(t) \|\mathbf{w}\|^2 + C_0(t) \|\mu\|^2, \\ \nu |\langle \nabla \rho^\nu, \mathbf{w} \rangle| &= \nu |\langle \nabla \mu, \mathbf{w} \rangle + \langle \nabla \rho^0, \mathbf{w} \rangle| \leq \frac{\nu |\sigma_1^{-1}|}{4} \|\nabla \mu\|^2 + \frac{2\nu + |\sigma_1^{-1}|}{2|\sigma_1^{-1}|} \|\mathbf{w}\|^2 + \frac{\nu^2}{2} \|\nabla \rho^0\|^2 \\ &\leq \frac{\nu |\sigma_1^{-1}|}{4} \|\nabla \mu\|^2 + \frac{2\nu + |\sigma_1^{-1}|}{2|\sigma_1^{-1}|} \|\mathbf{w}\|^2 + C_0(t) \nu^2. \end{aligned}$$

Now consider  $\langle \mu \eta, \Phi * \mu \rangle = \langle \mu \eta, \nabla \Phi * \mathbf{w} \rangle$  for  $d = 3$ . Write

$$\begin{aligned} |\langle \mu \eta, \nabla \Phi * \mathbf{w} \rangle| &\leq \|\mu \eta\|_{L^{6/5}} \|\nabla \Phi * \mathbf{w}\|_{L^6} \\ &\leq \|\mu\|_{L^2} \|\eta\|_{L^3} \|\nabla \Phi * \mathbf{w}\|_{L^6} \leq C_0(t) \|\mu\|_{L^2} \|\mathbf{w}\|_{L^2}, \end{aligned}$$

where we applied the Hardy-Littlewood-Sobolev Inequality. When  $\sigma_1 = -\sigma_2$ , the term  $\sigma_1(\sigma_1 + \sigma_2) \langle \mu \eta, \Phi * \mu \rangle$  disappears entirely.

Substituting these bounds into (6.11) and integrating in time gives, for all  $\nu \leq 1$ ,

$$\|\mathbf{w}(t)\|^2 \leq \frac{\nu}{2} \int_0^t \|\nabla \mu(s)\|^2 ds + C_0(t)t\nu + \int_0^t C_0(s) \left( \|\mathbf{w}(s)\|^2 + \|\mu(s)\|^2 \right) ds.$$

Adding this inequality to that in (6.8) gives, for all  $\nu \leq 1$ ,

$$\|\mathbf{w}(t)\|^2 + \|\mu(t)\|^2 + \frac{\nu}{2} \int_0^t \|\nabla \mu(s)\|^2 ds \leq C_0(t)t\nu + \int_0^t C_0(s) \left( \|\mathbf{w}(s)\|^2 + \|\mu(s)\|^2 \right) ds.$$

Applying Gronwall's lemma, we conclude that

$$\|\mathbf{w}(t)\|^2 + \|\mu(t)\|^2 + \nu \int_0^t \|\nabla \mu(s)\|^2 ds \leq C_0(t)t\nu e^{C_0(t)}$$

for all  $\nu \leq 1$ . The proof is completed by observing that  $\|\mathbf{w}(t)\|_{H^1} \leq \|\mathbf{w}(t)\| + C\|\mu(t)\|$  by the boundedness of Calderon-Zygmund operators on  $L^2$ .  $\square$

**Remark 6.3.** *An examination of the proof of Theorem 6.2 shows that the conclusion holds as long as the solutions satisfy  $\nabla \rho^0 \in L^\infty([0, T], L^2 \cap L^3(\mathbb{R}^d))$  when  $d = 3$ , and  $\nabla \rho^0 \in L^\infty([0, T], L^2 \cap L^q(\mathbb{R}^d))$  for some  $q > 2$  when  $d = 2$ , and  $\rho^\nu, \rho^0$  belong to  $L^\infty([0, T], L^1 \cap L^\infty(\mathbb{R}^d))$  for  $d = 2, 3$ . Our assumptions on the initial data imply that these conditions hold, but are not minimal; in particular, compact support of  $\rho_0$  is stronger than required.*

## 7. THE VANISHING VISCOSITY LIMIT FOR $(GAG_\nu)$ FOR VELOCITIES NOT IN $L^2$

In this section, we consider the vanishing viscosity limit  $(VV)'$  (see Section 1) in the general 2D case.

**Theorem 7.1.** *Assume that  $d = 2$  and  $\rho_0$  is compactly supported and in  $C_C^\alpha$  for some  $\alpha > 1$ . Define  $\mu, \mathbf{w}$  as in (6.1). Let  $T$  be as in Theorem 2.4. With  $\tau_0, g_0$  as in Definition 4.4 and the total mass function,  $m$ , defined as in (2.3), let*

$$\tilde{\mu} := \mu - m(\mu)g_0, \quad \tilde{\mathbf{w}} := \mathbf{w} - \sigma_1 m(\mu)\tau_0. \quad (7.1)$$

Then for all  $t \in [0, T]$ ,  $\nu \leq 1$ ,

$$\|\tilde{\mathbf{w}}(t)\|_{H^1}^2 + \|\mu(t)\|^2 + \nu \int_0^t \|\nabla \mu\|^2 \leq C\nu t e^{C_0(t)t}. \quad (7.2)$$

Moreover, for all  $k \geq 0$ ,

$$\|\mathbf{w}(t) - \tilde{\mathbf{w}}(t)\|_{C^k} \leq C_k \nu^{\frac{1}{2}} t^{\frac{3}{2}} e^{C_0(t)t}. \quad (7.3)$$

*Proof.* Assume that  $d = 2$ . Then  $\operatorname{div} \tilde{\mathbf{w}} = \sigma_1 \tilde{\mu}$ , and  $\tilde{\mathbf{w}} \in L^2(\mathbb{R}^d)$  for all time by Lemma 4.3, since  $m(\tilde{\mu}) = 0$ .

We start off the same way as in Section 6. Taking  $(GAG_\nu) - (GAG_0)$  gives

$$\partial_t \mu + \mathbf{w} \cdot \nabla \rho^0 + \mathbf{v}^\nu \cdot \nabla \mu = \sigma_2 \mu(\rho^0 + \rho^\nu) + \nu \Delta \rho^\nu.$$

We have  $\partial_t \mu, \Delta \rho^\nu \in L^2(0, T; H^{-1})$  while all other terms above lie in the smaller space,  $L^\infty(0, T; L^2)$ . Thus, we can take the pairing of the above equation with  $\varphi = \mu \in L^2(0, T; H^1)$ , giving

$$(\partial_t \mu, \mu) = -\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle - \langle \mathbf{v}^\nu \cdot \nabla \mu, \mu \rangle + \sigma_2 \langle \mu(\rho^0 + \rho^\nu), \mu \rangle + \nu \langle \Delta \rho^\nu, \mu \rangle, \quad (7.4)$$

as in Section 6, immediately following (6.2). Now, however, we need to estimate  $-\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle$  differently. First note that

$$|\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle| \leq |\langle \tilde{\mathbf{w}} \cdot \nabla \rho^0, \mu \rangle| + |\langle (\sigma_1 m(\mu)\tau_0) \cdot \nabla \rho^0, \mu \rangle|. \quad (7.5)$$

Following the proof of Theorem 6.2, we have

$$|\langle \tilde{\mathbf{w}} \cdot \nabla \rho^0, \mu \rangle| \leq \|\tilde{\mathbf{w}} \cdot \nabla \rho^0\| \|\mu\| \leq \|\tilde{\mathbf{w}}\|_{L^p} \|\nabla \rho^0\|_{L^q} \|\mu\| \quad (7.6)$$

where  $1/p + 1/q = 1/2$ . Now note that for  $p \in (2, \infty)$ , an argument identical to that in (6.4) yields

$$\|\tilde{\mathbf{w}}\|_{L^p} \leq C (\|\tilde{\mathbf{w}}\| + \|\tilde{\mu}\|).$$

Substituting this estimate into (7.6) gives, for any fixed  $q \in (2, \infty)$ ,

$$|\langle \tilde{\mathbf{w}} \cdot \nabla \rho^0, \mu \rangle| \leq C \|\nabla \rho^0\|_{L^q} (\|\tilde{\mathbf{w}}\| + \|\tilde{\mu}\|) \|\mu\| \leq C_0(t) (\|\tilde{\mathbf{w}}\| + \|\tilde{\mu}\|) \|\mu\|.$$

Now, by the definition of  $\tilde{\mu}$  and Propostion 6.1,

$$\begin{aligned} \|\tilde{\mu}(t)\| &\leq \|\mu(t)\| + |m(\mu(t))| \|g_0\| \leq \|\mu(t)\| + \|g_0\| |\sigma_1 + \sigma_2| \int_0^t \|\mu(s)\| \|\rho^\nu + \rho^0(s)\| ds \\ &\leq \|\mu(t)\| + C_0(t) \int_0^t \|\mu(s)\| ds, \end{aligned}$$

so that

$$\begin{aligned} |\langle \tilde{\mathbf{w}} \cdot \nabla \rho^0, \mu \rangle| &\leq C_0(t) \|\tilde{\mathbf{w}}\| \|\mu\| + C_0(t) \|\mu\|^2 + C_0(t) \|\mu\| \int_0^t \|\mu(s)\| ds \\ &\leq C_0(t) \|\tilde{\mathbf{w}}\|^2 + C_0(t) \|\mu\|^2 + \left( \int_0^t \|\mu(s)\| ds \right)^2 \\ &\leq C_0(t) \|\tilde{\mathbf{w}}\|^2 + C_0(t) \|\mu\|^2 + C_0(t) t \int_0^t \|\mu(s)\|^2 ds, \end{aligned} \quad (7.7)$$

where we used Jensen's inequality (or Cauchy-Schwartz) in the last step. To estimate  $|((\sigma_1 m(\mu) \tau_0) \cdot \nabla \rho^0, \mu)|$ , we observe that, by Proposition 6.1,

$$|m(\mu(t))| \leq |\sigma_1 + \sigma_2| \int_0^t (\|\rho^\nu(s)\| + \|\rho^0(s)\|) \|\mu(s)\| ds \leq C_0(t) \int_0^t \|\mu(s)\| ds \quad (7.8)$$

so that

$$\begin{aligned} |\langle (\sigma_1 m(\mu) \tau_0) \cdot \nabla \rho^0, \mu \rangle| &\leq |\sigma_1| \|\nabla \rho^0\| \|\tau_0\|_{L^\infty} |m(\mu)| \|\mu\| \\ &\leq \frac{1}{2} \sigma_1^2 \|\nabla \rho^0\|^2 \|\tau_0\|_{L^\infty}^2 m(\mu)^2 + \frac{1}{2} \|\mu\|^2 \leq C_0(t) \left( \int_0^t \|\mu(s)\| ds \right)^2 + \frac{1}{2} \|\mu\|^2 \\ &\leq C_0(t) t \int_0^t \|\mu(s)\|^2 ds + \frac{1}{2} \|\mu\|^2, \end{aligned} \quad (7.9)$$

where we again used Jensen's inequality (or Cauchy-Schwartz) as well as Proposition 6.1. Hence, applying (7.7) and (7.9) to (7.5),

$$|\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle| \leq C_0(t) \|\tilde{\mathbf{w}}\|^2 + C_0(t) \|\mu\|^2 + C_0(t) t \int_0^t \|\mu(s)\|^2 ds.$$

Integrating in time, we have

$$\int_0^t C_0(y) y \int_0^y \|\mu(s)\|^2 ds dy \leq C_0(t) t \int_0^t \int_0^t \|\mu(s)\|^2 ds dy \leq C_0(t) t^2 \int_0^t \|\mu(s)\|^2 ds. \quad (7.10)$$

Thus, in place of (6.8), we have

$$\|\mu(t)\|^2 + \nu \int_0^t \|\nabla \mu(s)\|^2 ds \leq C_0(t)t\nu + \int_0^t C_0(s) \left( \|\tilde{\mathbf{w}}(s)\|^2 + \|\mu(s)\|^2 \right) ds. \quad (7.11)$$

To bound  $\|\tilde{\mathbf{w}}\|$ , we derive the equivalent of (6.10) for  $\tilde{\mathbf{w}}$  for an arbitrary  $\varphi \in L^2(0, T; H^1)$ . The only change we need make is in (6.9)<sub>1</sub>, which, using  $\operatorname{div} \mathbf{w} = \operatorname{div} \tilde{\mathbf{w}} + \sigma_1 m(\mu)g_0$ , becomes

$$\begin{aligned} (\partial_t \mu, \varphi) &= \sigma_1^{-1} (\partial_t \operatorname{div} \mathbf{w}, \varphi) = \sigma_1^{-1} (\partial_t \operatorname{div} \tilde{\mathbf{w}}, \varphi) + m'(\mu)(g_0, \varphi) \\ &= -\sigma_1^{-1} \langle \partial_t \tilde{\mathbf{w}}, \nabla \varphi \rangle + (\sigma_1 + \sigma_2) m(\mu \eta)(g_0, \varphi), \end{aligned}$$

where  $\eta := \rho^0 + \rho^\nu$  and we used Proposition 6.1.

Thus, (6.10) becomes

$$\begin{aligned} -\sigma_1^{-1} \langle \partial_t \tilde{\mathbf{w}}, \nabla \varphi \rangle &= \langle \nabla \varphi \cdot \mathbf{w}, \rho^0 \rangle + \langle \nabla \varphi \cdot \mathbf{v}^\nu, \mu \rangle + (\sigma_1 + \sigma_2) \langle \mu \eta, \varphi \rangle \\ &\quad - \nu \langle \nabla \rho^\nu, \nabla \varphi \rangle - (\sigma_1 + \sigma_2) m(\mu \eta)(g_0, \varphi) \\ &= \langle \nabla \varphi \cdot \mathbf{w}, \rho^0 \rangle + \langle \nabla \varphi \cdot \mathbf{v}^\nu, \mu \rangle - \nu \langle \nabla \rho^\nu, \nabla \varphi \rangle + (\sigma_1 + \sigma_2) \langle \mu \eta - m(\mu \eta)g_0, \varphi \rangle. \end{aligned} \quad (7.12)$$

Now set  $\varphi = \sigma_1 \Phi * \tilde{\mu} = \sigma_1 \Phi * (\mu - m(\mu)g_0)$  and fix  $p_0 \in (2, \infty]$ . Then  $\nabla \varphi \in L^\infty(0, T; L^2)$  with  $\varphi \in L^\infty(0, T; L^{p_0})$  by Lemma 4.3. Also,  $\mu \eta \in L^\infty(0, T; L^{q_0})$ , where  $q_0$  is Hölder conjugate to  $p_0$ , by the rapid spatial decay of  $\mu$  and  $\eta$ . Hence, (7.12) holds for  $\varphi$  by an obvious density argument. Hence, using  $\nabla \varphi = \tilde{\mathbf{w}}$ ,

$$\begin{aligned} \sigma_1^{-1} \langle \partial_t \tilde{\mathbf{w}}, \tilde{\mathbf{w}} \rangle &= \frac{\sigma_1^{-1}}{2} \frac{d}{dt} \|\tilde{\mathbf{w}}\|^2 = -\langle \tilde{\mathbf{w}} \cdot \mathbf{w}, \rho^0 \rangle - \langle \tilde{\mathbf{w}} \cdot \mathbf{v}^\nu, \mu \rangle + \nu \langle \nabla \rho^\nu, \tilde{\mathbf{w}} \rangle \\ &\quad - \sigma_1 (\sigma_1 + \sigma_2) \langle \mu \eta - m(\mu \eta)g_0, \Phi * (\mu - m(\mu)g_0) \rangle. \end{aligned} \quad (7.13)$$

Now,

$$\begin{aligned} |\langle \tilde{\mathbf{w}} \cdot \mathbf{w}, \rho^0 \rangle| &\leq \|\rho^0\|_{L^\infty} \|\tilde{\mathbf{w}}\|^2 + |\sigma_1| m(\mu) |\langle \tau_0 \cdot \tilde{\mathbf{w}}, \rho_0 \rangle| \leq C_0(t) \|\tilde{\mathbf{w}}\|^2 + C_0(t) |m(\mu)|^2, \\ |\langle \tilde{\mathbf{w}} \cdot \mathbf{v}^\nu, \mu \rangle| &\leq \|\mathbf{v}^\nu\|_{L^\infty} \|\tilde{\mathbf{w}}\| \|\mu\| \leq C_0(t) \|\tilde{\mathbf{w}}\|^2 + C_0(t) \|\mu\|^2, \\ \nu |\langle \nabla \rho^\nu, \tilde{\mathbf{w}} \rangle| &= \nu |\langle \nabla \mu, \tilde{\mathbf{w}} \rangle + \langle \nabla \rho^0, \tilde{\mathbf{w}} \rangle| \leq \frac{\nu |\sigma_1^{-1}|}{4} \|\nabla \mu\|^2 + \frac{2\nu + |\sigma_1^{-1}|}{2|\sigma_1^{-1}|} \|\tilde{\mathbf{w}}\|^2 + \frac{\nu^2}{2} \|\nabla \rho^0\|^2 \\ &\leq \frac{\nu |\sigma_1^{-1}|}{4} \|\nabla \mu\|^2 + \frac{2\nu + |\sigma_1^{-1}|}{2|\sigma_1^{-1}|} \|\tilde{\mathbf{w}}\|^2 + C_0(t) \nu^2. \end{aligned}$$

Substituting these bounds into (7.13), integrating in time, using (7.8) with Jensen's inequality, and applying Lemma 7.2 we obtain, for all  $\nu \leq 1$ ,

$$\|\tilde{\mathbf{w}}(t)\|^2 \leq \int_0^t \frac{\nu}{2} \|\nabla \mu(s)\|^2 ds + C_0(t)t\nu + \int_0^t C_0(s) \left( \|\tilde{\mathbf{w}}(s)\|^2 + \|\mu(s)\|^2 \right) ds.$$

Adding this inequality to that in (7.11) gives, for all  $\nu \leq 1$ ,

$$\|\tilde{\mathbf{w}}(t)\|^2 + \|\mu(t)\|^2 + \frac{\nu}{2} \int_0^t \|\nabla \mu(s)\|^2 ds \leq C_0(t)t\nu + \int_0^t C_0(s) \left( \|\tilde{\mathbf{w}}(s)\|^2 + \|\mu(s)\|^2 \right) ds.$$

Applying Gronwall's lemma, we conclude that

$$\|\tilde{\mathbf{w}}(t)\|^2 + \|\mu(t)\|^2 + \nu \int_0^t \|\nabla \mu(s)\|^2 ds \leq C_0(t)t\nu e^{C_0(t)} \quad (7.14)$$

for all  $\nu \leq 1$ . Also, by Lemma 4.3 and Proposition 6.1,

$$\begin{aligned}
\|\tilde{\mathbf{w}}(t)\|_{H^1} &\leq \|\tilde{\mathbf{w}}(t)\| + C \|\tilde{\mu}(t)\| \leq \|\tilde{\mathbf{w}}(t)\| + \|\mu(t)\| + |m(\mu(t))| \|g_0\| \\
&\leq \|\tilde{\mathbf{w}}(t)\| + \|\mu(t)\| + \|g_0\| |\sigma_1 + \sigma_2| \int_0^t \|\mu(s)\| \|\eta(s)\| \, ds \\
&\leq \|\tilde{\mathbf{w}}(t)\| + \|\mu(t)\| + C_0(t) \int_0^t \|\mu(s)\| \, ds \\
&\leq \|\tilde{\mathbf{w}}(t)\| + C_0(t) (t\nu)^{\frac{1}{2}} e^{C_0(t)} + C_0(t) \int_0^t C_0(s) s^{\frac{1}{2}} \nu^{\frac{1}{2}} e^{C_0(s)} \, ds \\
&\leq \|\tilde{\mathbf{w}}(t)\| + C_0(t) (1+t)^{\frac{3}{2}} e^{C_0(t)} \nu^{\frac{1}{2}}.
\end{aligned}$$

In the second-to-last inequality, we used (7.14). Combining this bound with (7.14) completes the proof of (7.2).

To prove (7.3), we simply observe that

$$\begin{aligned}
\|\mathbf{w}(t) - \tilde{\mathbf{w}}(t)\|_{L^\infty} &= \|\sigma_1 m(\mu) \boldsymbol{\tau}_0\|_{L^\infty} \leq |\sigma_1| |m(\mu)| \|\boldsymbol{\tau}_0\|_{L^\infty} \leq C |m(\mu)| \\
&\leq C_0(t) \int_0^t \|\mu(s)\| \|\eta(s)\| \, ds \leq C \nu^{\frac{1}{2}} t^{\frac{3}{2}} e^{C_0(t)t},
\end{aligned}$$

where we used Proposition 6.1 and (7.2). A similar bound holds for all spatial derivatives of  $\mathbf{w}(t) - \tilde{\mathbf{w}}(t)$ , yielding (7.3).  $\square$

We used the following lemma in the proof of Theorem 7.1, above.

**Lemma 7.2.** *Define  $\mu$ ,  $\tilde{\mu}$ ,  $\tilde{\mathbf{w}}$  as in (6.1) and (7.1) and let  $\eta = \rho^0 + \rho^\nu$ , as in the proof of Theorem 7.1. When  $d = 2$ , we have,*

$$|\langle \mu\eta - m(\mu\eta)g_0, \Phi * \tilde{\mu} \rangle| \leq C_0(t) \|\tilde{\mathbf{w}}\| \|\mu\|.$$

*Proof.* First observe that  $\gamma := \mu\eta - m(\mu\eta)g_0$  lies in  $L^2(\mathbb{R}^2)$ , has total mass zero, and  $|x|^\varepsilon \gamma \in L^1(\mathbb{R}^2)$  for some  $\varepsilon \in (0, 1)$ . Thus, by Lemma 4.3,  $\nabla\Phi * \gamma \in L^2(\mathbb{R}^2)$ . Similarly,  $\tilde{\mu} \in L^2(\mathbb{R}^2)$  and  $\nabla\Phi * \tilde{\mu} = \tilde{\mathbf{w}} \in L^2(\mathbb{R}^2)$ . This allows us to integrate by parts, using  $\gamma = \operatorname{div}(\nabla\Phi * \gamma)$ , to conclude that

$$|\langle \gamma, \Phi * \tilde{\mu} \rangle| = |\langle \nabla\Phi * \gamma, \tilde{\mathbf{w}} \rangle| \leq \|\nabla\Phi * \gamma\| \|\tilde{\mathbf{w}}\|.$$

By Lemma 4.5, with  $a$  as in Definition 1.1,

$$\begin{aligned}
\|\nabla\Phi * \gamma\|^2 &= -\langle \Phi * \gamma, \gamma \rangle = -\langle (a\Phi) * \gamma, \gamma \rangle - \langle ((1-a)\Phi) * \gamma, \gamma \rangle \\
&\leq \|a\Phi\|_{L^1} \|\gamma\|^2 + |\langle ((1-a)\Phi) * \gamma, \gamma \rangle| \\
&\leq C_0(t) \|\mu\|^2 + |\langle ((1-a)\Phi) * \gamma, \gamma \rangle|,
\end{aligned} \tag{7.15}$$

since

$$\begin{aligned}
\|\gamma\|^2 &= \|\mu\eta - m(\mu\eta)g_0\|^2 \leq (\|\mu\| \|\eta\|_{L^\infty} + |m(\mu\eta)| \|g_0\|)^2 \\
&\leq (\|\mu\| \|\eta\|_{L^\infty} + \|\mu\| \|\eta\| \|g_0\|)^2 \leq C_0(t) \|\mu\|^2.
\end{aligned}$$

It remains to estimate  $|\langle ((1-a)\Phi) * \gamma, \gamma \rangle|$ . Define  $g(x) := 1 + |x|^\varepsilon$  and write,

$$\begin{aligned}
|\langle ((1-a)\Phi) * \gamma, \gamma \rangle| &= |\langle (1/g) [((1-a)\Phi) * \gamma], g\gamma \rangle| \\
&\leq \|(1/g) [((1-a)\Phi) * \gamma]\|_{L^\infty} \|g\gamma\|_{L^1}.
\end{aligned}$$

The compact support of  $\rho_0$  insures the compact support of  $\rho^0$  and, through Corollary 3.2, the rapid spatial decay of  $\rho^\nu$ . This allows us to conclude that  $\|g\eta\| \leq C_0(t)$ . Therefore,

$$\begin{aligned} \|g\gamma\|_{L^1} &= \|\mu g\eta - m(\mu\eta)gg_0\|_{L^1} \leq \|\mu\| \|g\eta\| + |m(\mu\eta)| \|gg_0\|_{L^1} \\ &\leq C_0(t) \|\mu\| + \|\mu\| \|\eta\| \|gg_0\|_{L^1} \leq C_0(t) \|\mu\|, \end{aligned}$$

and we conclude that

$$|\langle ((1-a)\Phi) * \gamma, \gamma \rangle| \leq C_0(t) \|(1/g) [((1-a)\Phi) * \gamma]\|_{L^\infty} \|\mu\|. \quad (7.16)$$

We need to extract another factor of  $\|\mu\|$  from  $\|(1/g) [((1-a)\Phi) * \gamma]\|_{L^\infty}$ . We have,

$$\begin{aligned} \left| \frac{1}{g(x)} [((1-a)\Phi) * \gamma](x) \right| &= \left| \frac{1}{2\pi g(x)} \int_{\mathbb{R}^2} (1-a(x-y)) \log|x-y| \gamma(y) dy \right| \\ &\leq \frac{1}{2\pi g(x)} \int_{\mathbb{R}^2} \frac{(1-a(x-y)) \log|x-y|}{|x-y|^\varepsilon} |x-y|^\varepsilon |\gamma(y)| dy \\ &\leq \frac{1}{2\pi g(x)} \int_{\mathbb{R}^2} \frac{(1-a(x-y)) \log|x-y|}{|x-y|^\varepsilon} (|x|+|y|)^\varepsilon |\gamma(y)| dy \\ &\leq \frac{1}{2\pi g(x)} \int_{\mathbb{R}^2} \frac{(1-a(x-y)) \log|x-y|}{|x-y|^\varepsilon} |x|^\varepsilon |\gamma(y)| dy \\ &\quad + \frac{1}{2\pi g(x)} \int_{\mathbb{R}^2} \frac{(1-a(x-y)) \log|x-y|}{|x-y|^\varepsilon} |y|^\varepsilon |\gamma(y)| dy. \end{aligned}$$

So

$$\begin{aligned} \left\| \frac{1}{g} [((1-a)\Phi) * \gamma] \right\|_{L^\infty} &\leq \left\| \frac{|x|^\varepsilon}{2\pi g(x)} \int_{\mathbb{R}^2} \frac{(1-a(x-y)) \log|x-y|}{|x-y|^\varepsilon} |\gamma(y)| dy \right\|_{L_x^\infty} \\ &\quad + \left\| \frac{1}{2\pi g(x)} \int_{\mathbb{R}^2} \frac{(1-a(x-y)) \log|x-y|}{|x-y|^\varepsilon} |y|^\varepsilon |\gamma(y)| dy \right\|_{L_x^\infty} \\ &\leq C \int_{\mathbb{R}^2} |\gamma(y)| dy + C \int_{\mathbb{R}^2} |y|^\varepsilon |\gamma(y)| dy = C \|\gamma\|_{L^1} + C \| |x|^\varepsilon \gamma(x) \|_{L_x^1}. \end{aligned} \quad (7.17)$$

But,

$$\begin{aligned} \|\gamma\|_{L^1} &= \|\mu\eta - m(\mu\eta)g_0\|_{L^1} \leq \|\mu\| \|\eta\| + |m(\mu\eta)| \|g_0\|_{L^1} \\ &\leq C_0(t) \|\mu\| + \|\mu\| \|\eta\| \|g_0\|_{L^1} \leq C_0(t) \|\mu\| \end{aligned}$$

and

$$\begin{aligned} \| |x|^\varepsilon \gamma(x) \|_{L_x^1} &= \|\mu(x)(|x|^\varepsilon \eta(x)) - m(\mu\eta)(|x|^\varepsilon g_0(x))\|_{L_x^1} \\ &\leq \|\mu\| \| |x|^\varepsilon \eta(x) \|_{L_x^2} + |m(\mu\eta)| \| |x|^\varepsilon g_0(x) \|_{L_x^1} \\ &\leq C_0(t) \|\mu\| + \|\mu\| \|\eta\| \| |x|^\varepsilon g_0(x) \|_{L_x^1} \leq C_0(t) \|\mu\|. \end{aligned}$$

Substituting this estimate into (7.17), the resulting estimate into (7.16), and finally that estimate into (7.15), yields the desired bound.  $\square$

## 8. THE VANISHING VISCOSITY LIMIT IN THE $L^\infty$ -NORM

In this section, we use the results from Section 6 and Section 7 to prove that the vanishing viscosity limit (see (VV) in Section 1) holds in the  $L^\infty$ -norm of the density. This follows immediately from interpolation if we are able to find a modulus of continuity on  $\rho^\nu$  that applies for all sufficiently small  $\nu$ . We do this by obtaining a bound on  $\rho^\nu$  in a Hölder

space norm *uniformly* in  $\nu$ , adapting the approach of Hmidi and Keraani in [12, 13] for the Navier-Stokes equations. This leads to the following theorem.

**Theorem 8.1.** *Assume  $\rho_0 \in C^\beta(\mathbb{R}^d)$  with  $\beta < 1$  and  $d \geq 2$ , and let  $T$  be as in Theorem 2.4. Then the smooth solution  $\rho^\nu$  to  $(GAG_\nu)$  belongs in  $L^\infty([0, T]; C^\beta(\mathbb{R}^d))$ , and the following estimate holds for all  $t \in [0, T]$ :*

$$\|\rho^\nu(t)\|_{C^\beta} \leq C_0(T) e^{e^{C_0(T)}}. \quad (8.1)$$

where  $C_0(T)$  depends on  $\|\rho_0\|_{C^\beta}$  and  $\beta$ .

*Proof.* That the solution  $\rho^\nu$  to  $(GAG_\nu)$  belongs in  $L^\infty([0, T]; C^\beta(\mathbb{R}^d))$  for all  $\beta > 0$  follows from standard arguments; we show only the uniform control in viscosity of the  $C^\beta$ -norm when  $\beta < 1$ .

This theorem is essentially proved in [12, 13] for divergence-free vector fields  $\mathbf{v}^\nu$  in the more general setting of Besov spaces. We follow the proof from [13] below, with a slight modification to account for the assumption that  $\mathbf{v}^\nu$  is not divergence-free in our setting. Our modification relies on a commutator estimate established in Chapter 4 of [7].

As in the proof in [13], we apply the Littlewood-Paley operator  $\Delta_j$  to  $(GAG_\nu)$  and apply the maximum principle (see, for example, Lemma 5 of [12]) to write

$$\|\Delta_j \rho^\nu(t)\|_{L^\infty} \leq \|\Delta_j \rho_0\|_{L^\infty} + C \int_0^t (\|[\Delta_j, \mathbf{v}^\nu \cdot \nabla] \rho^\nu(s)\|_{L^\infty} + \|\Delta_j (\rho^\nu(s))^2\|_{L^\infty}) ds.$$

Multiplying through by  $2^{j\beta}$  and taking the supremum over  $j$  gives

$$\begin{aligned} \|\rho^\nu(t)\|_{C^\beta} &\leq \|\rho_0\|_{C^\beta} + C \int_0^t \left( \sup_j 2^{j\beta} \|[\Delta_j, \mathbf{v}^\nu \cdot \nabla] \rho^\nu(s)\|_{L^\infty} + \|(\rho^\nu(s))^2\|_{C^\beta} \right) ds. \\ &\leq \|\rho_0\|_{C^\beta} + C \int_0^t \left( \sup_j 2^{j\beta} \|[\Delta_j, \mathbf{v}^\nu \cdot \nabla] \rho^\nu(s)\|_{L^\infty} + \|\rho^\nu(s)\|_{L^\infty} \|\rho^\nu(s)\|_{C^\beta} \right) ds, \end{aligned} \quad (8.2)$$

where we used the estimate  $\|(\rho^\nu)^2\|_{C^\beta} \leq C \|\rho^\nu\|_{L^\infty} \|\rho^\nu\|_{C^\beta}$  to obtain the last inequality. To bound the commutator on the right hand side in (8.2), we apply the commutator estimate

$$\|[\Delta_j, \mathbf{v}^\nu \cdot \nabla] \rho^\nu(s)\|_{L^\infty} \leq C 2^{-j\beta} \|\nabla \mathbf{v}^\nu(s)\|_{L^\infty} \|\rho^\nu(s)\|_{C^\beta}, \quad (8.3)$$

which is established in Chemin's proof of Lemma 4.1.1 of [7] (note that Chemin shows that (8.3) holds even when  $\operatorname{div} \mathbf{v}^\nu \neq 0$  and for all  $\beta > 0$ ). Substituting (8.3) into (8.2) and applying Gronwall's lemma gives

$$\|\rho^\nu\|_{L^\infty([0, T]; C^\beta)} \leq C e^{C V(t)} \|\rho_0\|_{C^\beta}, \quad (8.4)$$

where

$$V(t) = \int_0^t (\|\nabla \mathbf{v}^\nu(s)\|_{L^\infty} + \|\rho^\nu(s)\|_{L^\infty}) ds.$$

To complete the proof of Theorem 8.1, we apply Proposition 2.3.5 of [7] and write

$$\|\nabla \mathbf{v}^\nu(t)\|_{L^\infty} \leq \frac{C}{\beta} \|\nabla \mathbf{v}^\nu(t)\|_{C_*^0} \log \left( e + \frac{\|\nabla \mathbf{v}^\nu(t)\|_{C^\beta}}{\|\nabla \mathbf{v}^\nu(t)\|_{C_*^0}} \right).$$

Since  $x \mapsto x \log(e + \frac{C}{x})$  is increasing in  $x$  when  $C > 0$ , it follows from Lemma 8.3, the embedding  $L^\infty \hookrightarrow C_*^0$ , and the equivalence between  $C^\beta$  and  $C_*^\beta$  when  $\beta \in (0, 1)$  that

$$\begin{aligned} \|\nabla \mathbf{v}^\nu(t)\|_{L^\infty} &\leq \frac{C}{\beta} \|\rho^\nu\|_{L^1 \cap L^\infty} \log \left( e + \frac{\|\nabla \mathbf{v}^\nu(t)\|_{C^\beta}}{\|\rho^\nu\|_{L^1 \cap L^\infty}} \right) \leq \frac{C}{\beta} \|\rho^\nu\|_{L^1 \cap L^\infty} \log \left( e + \frac{\|\rho^\nu(t)\|_{L^1 \cap C^\beta}}{\|\rho^\nu\|_{L^1 \cap L^\infty}} \right) \\ &\leq \frac{C}{\beta} C_0(T) \log \left( e + \frac{\|\rho^\nu(t)\|_{L^1 \cap C^\beta}}{C_0(T)} \right), \end{aligned}$$

where we used Theorem 2.4 in the third inequality above. An application of 8.4 gives

$$\begin{aligned} \|\nabla \mathbf{v}^\nu(t)\|_{L^\infty} &\leq C_0(T) \log \left( e + C e^{C V(t)} \|\rho_0\|_{C^\beta} \right) \\ &\leq C_0(T) \log \left( e^{C V(t)} (e + \|\rho_0\|_{C^\beta}) \right) = C_0(T) (V(t) + \log(e + \|\rho_0\|_{C^\beta})) \\ &= C_0(T) \log(e + \|\rho_0\|_{C^\beta}) + C_0(T) \int_0^t (\|\nabla \mathbf{v}^\nu(s)\|_{L^\infty} + \|\rho^\nu(s)\|_{L^\infty}) ds. \end{aligned}$$

Gronwall's lemma and Theorem 2.4 imply that

$$\|\nabla \mathbf{v}^\nu(t)\|_{L^\infty} \leq C_0(T) e^{C_0(T)}.$$

Substituting this estimate into (8.4) gives (8.1).  $\square$

We now prove the main result of this section.

**Theorem 8.2.** *Assume  $\rho_0$  is compactly supported and belongs to  $C^\alpha(\mathbb{R}^d)$  with  $\alpha > 1$ ,  $d \geq 2$ . Define  $\mu$  as in (6.1), and let  $T$  be as in Theorem 2.4. Then for  $t \in [0, T]$ ,  $\nu \leq 1$ , and  $\beta \in (0, 1)$ ,*

$$\|\mu(t)\|_{L^\infty} \leq C_0(t) (\nu t)^{\frac{2\beta}{2\beta+d}}.$$

*Proof.* Fix  $t < T$ ,  $p > d$ , and  $\beta \in (0, 1)$ . For fixed  $N \geq 0$  (to be chosen later), we use Bernstein's Lemma and the definition of the Holder space  $C^\beta(\mathbb{R}^d)$  as given in Definition 4.2 to write

$$\begin{aligned} \|(\rho^\nu - \rho^0)(t)\|_{L^\infty} &\leq \sum_{j=-1}^N \|\Delta_j(\rho^\nu - \rho^0)(t)\|_{L^\infty} + \sum_{j=N+1}^\infty \|\Delta_j(\rho^\nu - \rho^0)(t)\|_{L^\infty} \\ &\leq C \sum_{j=-1}^N 2^{j\frac{d}{2}} \|\Delta_j(\rho^\nu - \rho^0)(t)\|_{L^2} + \sum_{j=N+1}^\infty 2^{-j\beta} 2^{j\beta} \|\Delta_j(\rho^\nu - \rho^0)(t)\|_{L^\infty} \\ &\leq C \sum_{j=-1}^N 2^{j\frac{d}{2}} \|\Delta_j(\rho^\nu - \rho^0)(t)\|_{L^2} + C \sum_{j=N+1}^\infty 2^{-j\beta} (\|\rho^\nu(t)\|_{C^\beta} + \|\rho^0(t)\|_{C^\beta}) \\ &\leq C 2^{N\frac{d}{2}} \|(\rho^\nu - \rho^0)(t)\|_{L^2} + C_0(t) 2^{-N\beta}, \end{aligned} \tag{8.5}$$

where we applied Theorems 5.2 and 8.1 above to get the last inequality. By Theorems 6.2 and 7.1, for  $\nu \leq 1$ ,

$$\|(\rho^\nu - \rho^0)(t)\|_{L^2} \leq C_0(t) t \nu e^{C_0(t)}.$$

Substituting this estimate into (8.5) gives

$$\|(\rho^\nu - \rho^0)(t)\|_{L^\infty} \leq C_0(t) t e^{C_0(t)} \nu 2^{N\frac{d}{2}} + C_0(t) 2^{-N\beta}.$$

Now set  $N = -\frac{2}{2\beta+d} \log_2(\nu t)$ . We conclude that

$$\|(\rho^\nu - \rho^0)(t)\|_{L^\infty} \leq C_0(t) t e^{C_0(t)} (\nu t)^{1-\frac{d}{2\beta+d}} + C_0(t) (\nu t)^{\frac{2\beta}{2\alpha+d}} \leq C_0(t) e^{C_0(t)} (\nu t)^{\frac{2\beta}{2\beta+d}}.$$



This completes the proof of Theorem 8.2.  $\square$

Above, we used the following lemma.

**Lemma 8.3.** *For all  $r \in \mathbb{R}$ ,*

$$\|\nabla \nabla \Phi * \rho\|_{C_*^r} \leq C(\|\rho\|_{L^1} + \|\rho\|_{C_*^r}).$$

*Proof.* Let  $\mathbf{v} = \nabla \Phi * \rho$ . Then using the definition of  $C_*^r$  as given in Definition 4.2, we have

$$\begin{aligned} \|\nabla \mathbf{v}\|_{C_*^r} &= \sup_{q \geq -1} 2^{qr} \|\Delta_q \nabla \mathbf{v}\|_{L^\infty} \leq 2^{-r} \|\Delta_{-1} \nabla \mathbf{v}\|_{L^\infty} + \sup_{q \geq 0} 2^{qr} \|\Delta_q \nabla \mathbf{v}\|_{L^\infty} \\ &\leq C \|\Delta_{-1} \rho\|_{L^1} + C \sup_{q \geq 0} 2^{qr} \|\Delta_q \rho\|_{L^\infty} \leq C \|\rho\|_{L^1} + C \|\rho\|_{C_*^r}. \end{aligned}$$

To obtain the second inequality above, we argued as in (3.6) of [23] when estimating the low frequency term and applied a classical lemma to bound the high frequencies (see, for example, Lemma 4.2 of [10]). This completes the proof.  $\square$

## 9. CONCLUDING REMARKS

It is possible to obtain a velocity formulation of  $(GAG_\nu)$ , in analogy with the Navier-Stokes and Euler equations. For any  $\nu \geq 0$ , we can write this in the form,

$$\begin{cases} \partial_t \mathbf{v}^\nu + \mathbf{v}^\nu \cdot \nabla \mathbf{v}^\nu + \nabla q^\nu = \nu \Delta \mathbf{v}^\nu, \\ \operatorname{curl} \mathbf{v}^\nu = 0, \\ \mathbf{v}^\nu(0) = \mathbf{v}_0, \end{cases}$$

for an appropriately chosen “pressure,”  $q^\nu$ . This velocity formulation can be used to obtain the bounds on  $\|\mathbf{w}(t)\|$  in Section 6 and  $\|\tilde{\mathbf{w}}(t)\|$  in Section 7. Because  $\operatorname{div} \mathbf{w} \neq 0$ , however, the pressure does not disappear in these bounds. This requires a great deal of effort to properly bound the pressure, so we took the shorter approach in Sections 6 and 7, leaving the elaboration of the velocity formulation to future work.

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